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A LEVEL MATHS
YEAR 2 PURE

This book started off as homework sheets to encourage my students to do work at home straight after the lesson we had just done. It ended up turning into this, a book, one intended to help guide you through the A Level Maths course with questions of all difficulties.

The questions aren't C/D or A/A* questions as such, just the type of skill level who is working at that grade.

The questions have evolved as the exams have appeared on the 'new spec' rather than being ill fitting old questions from previous iterations of the course.

I have been a qualified maths teacher teaching A Level and Further A Level Maths for 15 years and hope this book makes me millions so I can retire to a fishing village in Scotland with my family and dogs. I also hope this book helps you or a loved one become just a little more confident.

Thanks to my parents for being amazing for the last 45 years.

Thanks to my partner for helping me to live the life I want to.

Thanks to my students who have (hopefully) proofread my questions and answers. That said, some minor errors may have slipped through!

Thank you to the Whippets and Iz for forcing me to get out every day and not being a socially awkward hermit.

Thanks to the students of Spalding Grammar School for their corrections, edits and the like!

I can’t write a book without showcasing my brother’s artwork. He works out of his studio and gallery (The Point in Cromer). His work is beautiful and can be found at www. richardkbladesartist.co.uk

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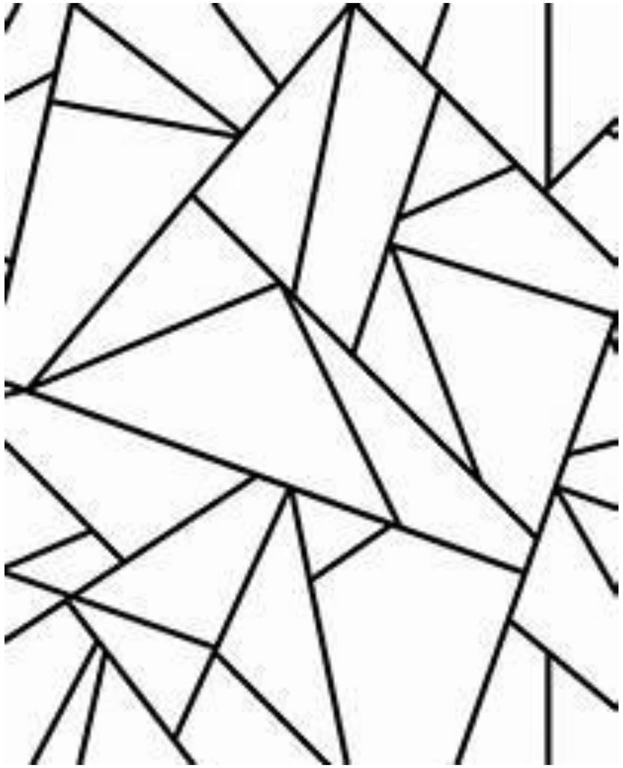
Algebra



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(1) Proof by Contradiction

WORKING AT D/E

(1) Prove, by contradiction, that the curves with equations $y = x^4 + 10x^2$ and $y = 2x^2 - 15$ for $x \epsilon R$ have no points of intersection.

(2) Prove, by contradiction, that there is no largest multiple of 17.

WORKING AT B/C

(1) Prove, by contraction, that there is no largest integer.

(2) Prove by contradiction, that if n^3 is odd then n is odd for $n \in Z^+$

(3) Prove by contradiction, that if n^2 is odd then n is odd for $n \in Z^+$

(4) Prove by contradiction that $\sqrt{2}$ is irrational number.

WORKING AT A*/A

(1) Prove, by contradiction that if a and b are such that $a, b \in Z^+$, then $a^2 - 4b \neq 2$

(2) Prove, by contradiction, that n and $n + 1$, for $n \in Z^+, n > 1$ have no common prime factors

(3) Prove, by contradiction, that there are infinitely many prime numbers.

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(2) Algebraic Fractions (Simplifying)

WORKING AT D/E

(1) Find each of the following as a single, simplified fraction:

(a) $\frac{x^2}{(x+1)} \times \frac{2x+2}{x}$

(b) $\frac{x^2-1}{3y(x+1)} \times \frac{y^2}{(x-1)}$

(c) $\frac{x^2-36}{3y-3} \times \frac{y^2-1}{(x-6)}$

(d) $\frac{x^2-3x-10}{3(x+2)} \div \frac{(x-5)}{(x-4)}$

(e) $\frac{10x-10}{y} \div \frac{x^2-1}{y^2-y}$

(f) $\frac{(x+3)(x-2)}{x^2-3x} \times \frac{x^3}{(x\mp 2)}$

(2) Find each of the following as a single, simplified fraction:

(a) $\frac{2}{x} + \frac{1}{(x+1)}$

(b) $\frac{4}{(x-1)} + \frac{3}{(x+1)}$

(c) $\frac{10}{x^2} - \frac{2}{(x+3)}$

(d) $\frac{5}{x} + \frac{1}{x^2}$

(e) $\frac{9}{x^2-x-6} - \frac{1}{(x-3)}$

(f) $\frac{2}{(x-7)} + \frac{x}{(x+1)}$

(3) Cyril wants to simplify the fraction $\frac{x+7}{x+14}$. Advise him on what he can do.

WORKING AT B/C

(1) Find each of the following as a single, simplified fraction:

(a) $\frac{x^2+x}{6x+6} \times \frac{25x^2-36}{5x+6}$

(b) $\frac{x^2-1}{x^2+x-2} \times \frac{y^2-10y}{y}$

(c) $\frac{x^3-x^2}{3y^2-2y-1} \times \frac{y^2-1}{14-14x}$

(d) $\frac{6x^2+5x-6}{9x-6} \div \frac{4x+6}{9}$

(e) $\frac{2-x}{14x+2} \times \frac{49x^2-1}{x^2-2x}$

(f) $\frac{2x^4-2x^3}{x^2-2x-8} \div \frac{4x^3}{(x\mp 2)}$

(2) Find each of the following as a single, simplified fraction:

(a) $\frac{1}{x^2-3x-10} + \frac{1}{x^2+x-30}$

(b) $\frac{1}{x^2} - \frac{3}{x(x+3)}$

(c) $\frac{5}{x^3} + \frac{1}{x^2}$

(d) $\frac{3}{x^2} + \frac{4}{(x+1)} - \frac{1}{x}$

(3) Show that $2 + \frac{3}{x-1} - \frac{10}{x^2-1} \equiv \frac{(2x-3)(x+3)}{x^2-1}$

WORKING AT A*/A

(1) Simplify $\frac{2x^4-2}{x^2+1} \div \frac{x-1}{4x}$

(2) Simplify $\frac{3-x}{x^2-3x} \times \frac{5x^3-5x^2}{6x^2+4x-2}$

(3) (a) Write $-2 + \frac{a}{b} + \frac{b}{a}$ as a fully simplified single fraction.

(b) Doris is asked to solve the equation $-2 + \frac{a}{b} + \frac{b}{a} = 0$ where a and b are real numbers.

What statement can Doris make about a and b? You must justify your answer.

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(3) Partial Fractions

WORKING AT D/E

(1) Show that $\frac{6x-2}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)}$
where A and B are integers to be found.

(2) Express each of the following in partial fractions

(a) $\frac{6x+18}{(x+4)(x-2)}$

(b) $\frac{x+15}{(x-6)(x+1)}$

(c) $\frac{3x-35}{x(x-7)}$

(3) (a) Factorise $x^2 - 2x - 24$

(b) Hence, express $\frac{5x}{x^2-2x-24}$ in partial fractions

WORKING AT B/C

(1) Express $\frac{5x+21}{(x+6)(4x+6)}$ in partial fractions.

(2) Express each of the following in partial fractions

(a) $\frac{7x-9}{x^2-x}$

(b) $\frac{-12x-20}{36x^2-25}$

(c) $\frac{5(2x-1)}{6x^2+x-1}$

(3) Show that $\frac{13x-4}{15(x^2+x-2)} \equiv \frac{A}{(x+2)} + \frac{B}{(x-1)}$
where A and B are rational fractions to be found.

WORKING AT A*/A

(1) Show, **<u>using partial fractions</u>** that

$$\frac{-2x^3 - x^2 - 2x + 7}{x^4 - 1}$$

can be written as $-\frac{4}{x^2+1} + \frac{1}{2(x-1)} - \frac{5}{2(x+1)}$

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(4) Partial Fractions with Repeated Factors

WORKING AT D/E

(1) Show that $\frac{3x+2}{x^2(x+1)}$ can be written in the form $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x+1)}$ where A, B and C are integers.

(2) Using partial fractions find the values of A, B and C, given that

$$\frac{5x^2+13x+5}{x(x+1)^2}=\frac{A}{x}+\frac{B}{(x+1)}+\frac{C}{(x+1)^2}$$

(3) Express $\frac{8x^2-27x+20}{x(x-2)^2}$ in the form $\frac{A}{x}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}$

WORKING AT B/C

(1) Express $\frac{-16x^2+19x+1}{(x-1)(2x-1)^2}$. You must show full workings.

(2) (a) Factorise x^3-3x^2

(b) Hence, show that $\frac{-2x^2-11x+15}{x^3-3x^2}$ can be written in partial fractions

WORKING AT A*/A

(1) Express $\frac{-x^2-4x+15}{x^3-3x^2+4}$ in partial fractions

(2) Express $\frac{2x^2-4x+3}{(x-1)^3}$ in partial fractions

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(5) Partial Fractions Requiring Algebraic Division

WORKING AT D/E

(1) Circle **all** of the fractions below that are improper.

$\frac{x^2+2x+1}{2x^2-3x-7}$ $\frac{x^3-x}{x^4+x+12}$ $\frac{x^2+6x+2}{x-5}$ $\frac{3x^2-x}{x(x^2+9)}$

(2) Show that

$$\frac{2x^2-2x-16}{(x+1)(x-3)} \equiv A + \frac{B}{(x+1)} + \frac{C}{(x-3)}$$

where A, B and C are integers to be found.

(3) Use algebraic division to show that

$$\frac{x^3-x^2-17x+20}{(x-4)}$$

Can be written in the form $(x-4) \times f(x)$

WORKING AT B/C

(1) (a) Use polynomial division to show that $\frac{3x^2+22x+6}{x^2+3x-18}$ can be written as:

$$3 + \frac{13x+60}{(x+6)(x-3)}$$

(b) Hence, express $\frac{3x^2+20x+12}{x^2+3x-18}$ in the form

$$3 + \frac{A}{(x+6)} + \frac{B}{(x-3)}.$$

(2) (a) Use polynomial division to show that

$$\frac{3x^2-23x+32}{x^2-5x-14}$$

can be written in the form $A + \frac{74-8x}{(x-7)(x+2)}$

(b) Hence, express $\frac{3x^2-23x+32}{x^2-5x-14}$ in partial fractions

WORKING AT A*/A

(1) Express:

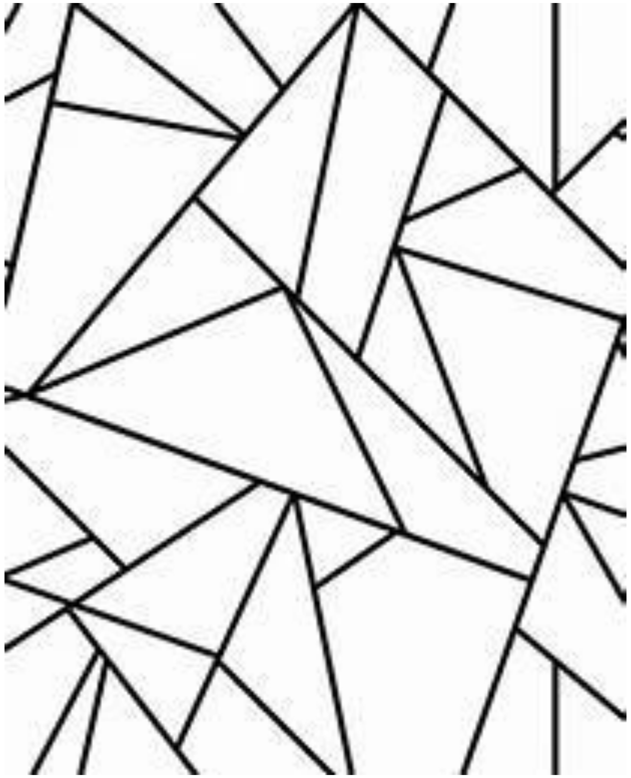
$$\frac{4x^3+x^2+2x-2}{x^2(x-1)}$$

In partial fractions. You must show full workings.

(2) Express $\frac{9x^2+1}{9x^2-1}$ in partial fractions.

You must show full workings.

Functions



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(6) Introduction to the Modulus Function

WORKING AT D/E

(1) $g(x) = |-x^2 - x|$

(a) Find $g(2)$ (b) Find $g(-1)$

(c) Write down the number of solutions to the equation $g(x) = -6$

(2) Sketch the graph of each showing where the line meets or crosses the coordinate axes.

(a) $y = |x|$ (b) $y = |x - 3|$ (c) $y = |2x - 1|$

(3) Find the two solutions to the equation $|3x - 1| = x + 2$.

WORKING AT B/C

(1) (a) Sketch the graphs of $y = |ax - b|$ where a and b are positive constants showing where the line meets or crosses the coordinate axes in terms of a and b .

(b) Show that the solution to the equation

$|ax - b|$ = 3 can be written as $x = \frac{b \pm 3}{a}$

(2) (a) Solve the equation $|2x + 5| = 3 - x$

(b) Hence, solve the inequality $|2x + 5| < 3 - x$

(3) Solve the equation $|2x - 1| = |3 - x|$

WORKING AT A*/A

(1) There are no solutions to the equation

$$|4x - 1| = mx$$

where m is a constant. Write down the possible set of values of m.

(2) (a) The equation $|px + q| = r$ has two real solutions. Write down the set of values of the constant r.

(b) Sketch the graph of $y = |px + q| - r$ where $p > 0$, $r > 0$ and $q < 0$. Show where the graph meets or crosses the coordinate axes and the minimum point on the graph in terms of p, q and r.

(3) $f(x) = Ax^2 + Bx + C$ and $g(x) = a$

Find the maximum possible number of solutions to the equation $|f(x)| = g(x)$

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(7) Mappings and Functions

WORKING AT D/E

(1) $f(x) = x^2 - 3,\ x \in R,\ x > 0$
(a) **<u>Write down</u>** the domain of the function $f(x)$
(b) Sketch the graph of $y = f(x)$.
(c) Hence, state the range of $f(x)$.
(d) What type of mapping is $f(x)$?
Pick one of the 4 choices given below:

Not a function 1-2-1	A Function, Many to one
Not a function, Many to Many	A Function, 1-2-1

(e) Given that $f(a) = 22$, find the value of a and explain why there is only one possible value of a.

(2) $f: (x) \rightarrow 3x - 1,\ x \in R$
(a) Write down the range of $f(x)$
(b) Find the value of p such that $f(p) = -13$

(3) $g(x) = \begin{cases} x^2, & 0 \leq x \leq 2 \\ x + 2, & 2 < x < 10 \end{cases}$
(a) Sketch the graph of $y = g(x)$
(b) Find the range of $g(x)$
(c) Solve the equation of $g(x) = 1$

WORKING AT B/C

(1) $h(x) = e^x - 6,\ x \in R$
(a) Sketch the graph of $y = h(x)$ showing the exact values where the curve meets the coordinate axes and write down the equation of the asymptote.
(b) Write down the range of $h(x)$
(c) Given that $h(a) = 2$, find the exact value of a.
(d) Sketch the graph of $y = |h(x)|$ stating its range.

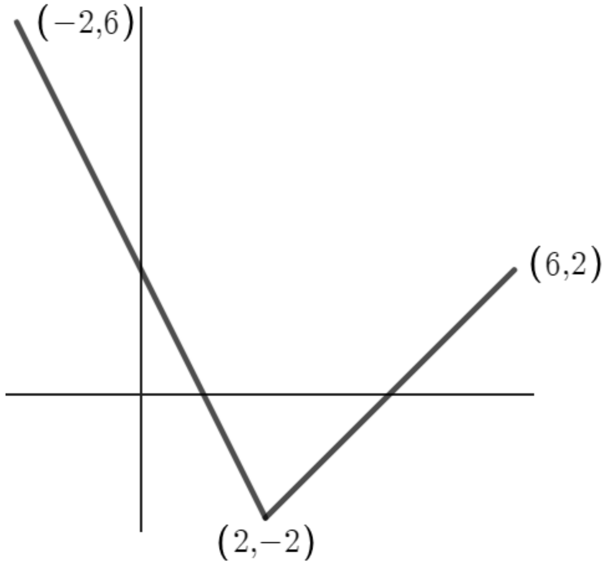
(2) $f(x) = x^2 - 2x + 10,\ x \in R$
(a) Write $f(x)$ in the form $(x + p)^2 + q$
(b) Sketch the graph of $y = f(x)$
(c) Explain why $f(x)$ is not a 1-2-1 function.
(d) Find a suitable domain that makes $f(x)$ a 1-2-1 function.
Doris choses the domain $0 > x$ for $f(x)$,
(e) Using Doris's domain, solve the equation $f(x) = 45$.
(f) Explain why the graph of $y = f(x)$ and the graph of $y = |f(x)|$ look the same.

(3) $g(x) = \frac{4}{x^2},\ 1 < x < 4$
(a) Sketch the graph of $y = g(x)$
(b) Find the range of $g(x)$
(c) Explain why $g(x) \neq \frac{37}{4}$

WORKING AT A*/A

(1) $f(x) = 2x^3 + 3x^2 - 12x,\quad -3 \leq x \leq 1$
(a) Find the coordinate of any stationary points on the graph of $y = f(x)$.
(b) Hence, find the range of $f(x)$.
(c) Find the exact solutions to the equation $f(x) = 0$
(d) The equation $f(x) + a = 0$ where a is a constant, has no solutions. Find the possible set of values of a.

(2) The function $h(x)$ is shown below.



$h(x)$ is linear and piecewise.
(a) Write a possible expression for the function $h(x)$ including the domain.
(b) Write down the range of $h(x)$
(c) Find $h(2)$
(c) Solve the equation $h(x) = 1$

(3) Find the range of $m(x) = 12 - e^x,\ -1 < x < 4$ giving your answer in exact form.

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(8) Composite Functions

WORKING AT D/E

(1) $f(x) = (x+1)^2, x \in R$ and $g(x) = 2x - 1, x \in R$

(a) Find (i) $gf(2)$ (ii) $fg(-4)$ (iii) $f^2(-2)$

(b) Show that $fg(x) \equiv Ax^2$ where A is a constant to be found.

(c) Show that $gf(x) \equiv 2x^2 + 4x + 1$

(d) Hence, solve the equation $gf(x) = 1$

(2) $f: (x) \to \frac{1}{x},\ x > 0$ and $g: (x) \to x^2,\ x \in R$

(a) Find a simplified expression for $ff(x)$

(b) Hence, solve the equation $ff(x) = gf(x)$

(3) $f(x) = 4x + 1$

Show that $f^2(x) \neq [f(x)]^2$

WORKING AT B/C

(1) $f(x) = e^{2x}, x \in R$ and $g(x) = \ln(3x - 1), x \in R, x > \frac{1}{3}$

(a) Show that $fg(x)$ can be written in the form $(Ax + B)^2$ where A and B are integers to be stated.

(b) Hence, solve the equation $fg(x) = 25$.

(c) Explain why there is only one solution to $fg(x) = 25$

(d) Find the exact solution to the equation

$$gf(x) = ln8$$

(2) $h(x) = \frac{2}{x-3},\ x \in R,\ x \neq 3$

(a) Show that $h^2(x) = \frac{2x-6}{11-3x}$

(b) Hence, solve the equation $h^2(x) = h(x)$ giving your answers in exact form.

(3) $f(x) = 4 - x,\ x \in R$ and $g(x) = |x|,\ x \in R$

(a) Sketch the graph of $y = gf(x)$

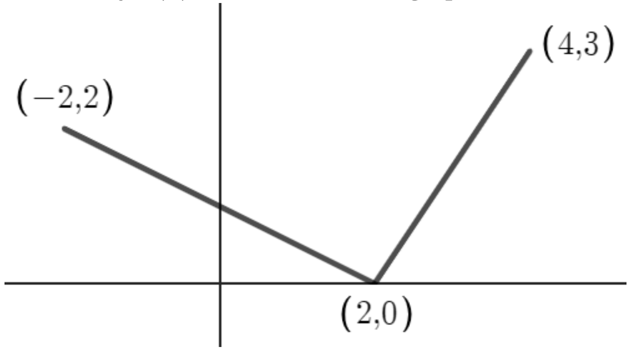
(b) Sketch the graph of $y = fg(x)$

WORKING AT A*/A

(1) $m(x) = 3^x,\ x \in R$ and $n(x) = 2x,\ x \in R$

Solve the equation $mn(x) = n(3) - m(x)$ giving your answer in exact form.

(2) The graph of the linear piecewise function $t(x), -2 \leq t \leq 4$ is shown below. 3 of the points that satisfy $t(x)$ are shown on the graph.



A second function $s(x)$ is such that $s(x) = x^3, x \in R$. Solve the equation $st(x) = 8$

(3) $h(x) = \frac{1}{x}, x \in R, x \neq 0$ and $g(x) = x - 4$

(a) Show that $h(x)$ is self-inverting

(b) Sketch the graphs of $y = hg(x)$ and $y = gh(x)$ on the same set of axes

(c) Hence, solve the equation $hg(x) = gh(x)$ giving your answers in exact form.

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(9) Inverse Functions

WORKING AT D/E

(1) $f(x) = (x+2)^2, x \in R,\ x \geq -2$
(a) Write down the domain of $f(x)$
(b) Sketch the graph of $y = f(x)$
(c) Hence state the range of $f(x)$
(d) Show that the inverse function $f^{-1}(x) = \sqrt{x} - 2$
Using your answers to parts (a), (b) and (c)
(e) Sketch the graph of $y = f^{-1}(x)$ stating the domain the range of $f^{-1}(x)$
(f) Find $f^{-1}(1)$

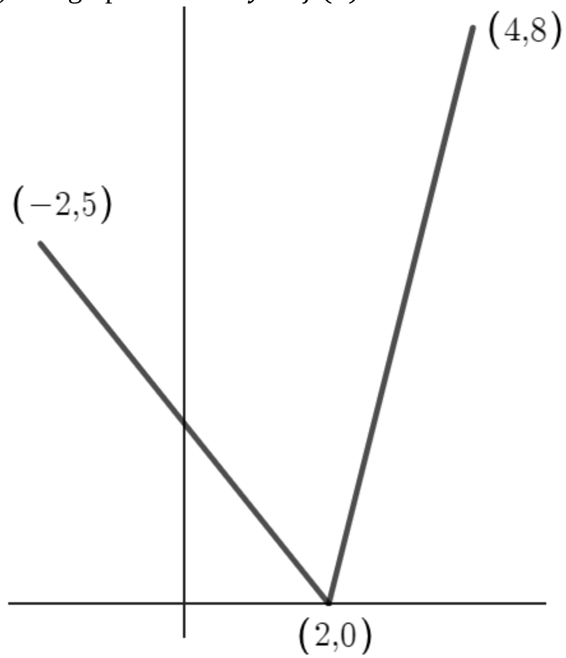
(2) $g(x) = x^3 - 3,\ x \in R,\ x \geq 0$
(a) Find $g^{-1}(x)$ stating its domain and range.
(b) Sketch the graph of $y = g^{-1}(x)$

(3) $f(x) = x^2 - 2x - 3,\ x \in R$
Explain why it is not possible to find $f^{-1}(x)$

WORKING AT B/C

(1) $h(x) = \sqrt[3]{x+6},\ x \in R, x \geq -6$
(a) State the domain of $h(x)$
(b) Find the range of $h(x)$
(c) Find an expression for $h^{-1}(x)$
(d) State the domain and range of $h^{-1}(x)$.
(e) Solve the equation $h^{-1}(x) = h(x)$

(2) The graph below is $y = f(x)$

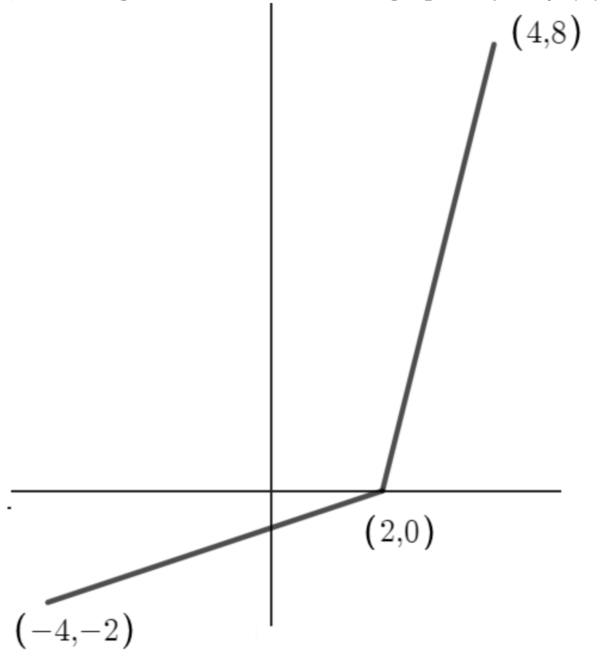


(a) Explain why it's not possible to find $f^{-1}(x)$
The domain of $f(x)$ is now restricted to $2 \leq x \leq 3$
(b) Sketch the graph of $y = f^{-1}(x)$ stating its domain and range.

(3) $f: (x) \to e^{2x} + 1, x \in R$.
Find the inverse function stating its domain and range.

WORKING AT A*/A

(1) The diagram below shows the graph of $y = f(x)$



$f(x)$ is made up of two linear parts for $-4 \leq x \leq 4$
$g(x) = x(x-3), x \in R, x > 0$
Solve the equation $fg(x) = 8$

(2) $h(x) = x^2 + 2px + q$ where p and q are positive constants.
Cyril suggests $h^{-1}(x) = -p + \sqrt{x + p^2 + q}$
Could he be correct?

(3) $t(x) = \frac{1}{x-1}, x \in R, x \neq 1$ and $s(x) = 8$
Solve the equation $s^{-1}t^{-1}(x) = x$

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(10) The Functions $y = |f(x)|$ and $y = f(|x|)$

WORKING AT D/E

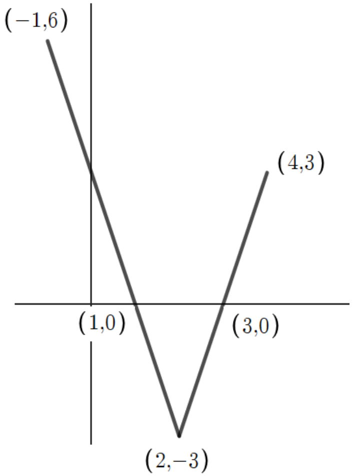
(1) $f(x) = x^2 - x - 6,\ x \in R$

(a) Sketch the graph of $y = f(x)$ showing where the graph crosses the coordinate axes.

(b) Hence, sketch the graph of each of the following showing where the graph meets or crosses the coordinate axes: (i) $y = |f(x)|$ (ii) $y = f(|x|)$

(c) State **the number** of solutions to the equation $|x^2 - x - 6| = -3$

(2) The graph of $y = g(x)$ is shown below.



On separate diagrams, sketch the graphs of (i) $y = |g(x)|$ and (ii) $y = f(|x|)$, stating points where the graph meets the coordinate axes and the coordinates of any turning points.

WORKING AT B/C

(1) (a) Sketch the graph of $y = e^{|x|}$ showing where the graph crosses the coordinate axes.

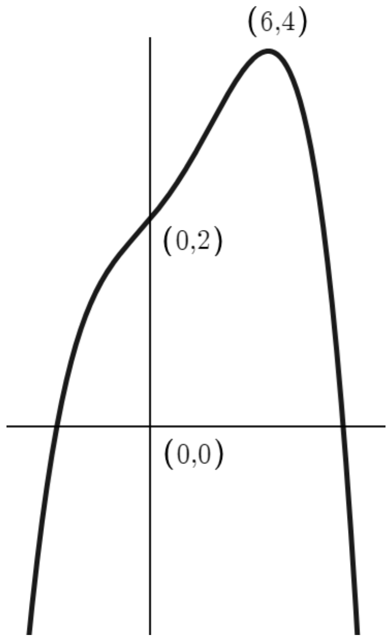
(b) Find the exact solutions to the equation $e^{|x|} = 4$

(2) (a) Sketch the graph of $y = |\ln(x-5)|$ showing where the graph meets the x axis and writing down the equation of the vertical asymptote.

(b) Solve the equation $2 = |\ln(x-5)|$ giving your answers in exact form.

(c) Sketch the graph of $y = \ln(|x|)$ labelling the equation of the asymptote.

(3) The graph of $y = g(x)$ is shown below



(a) How many solutions are there to the equation $|g(x)| = 1$?

(b) How many solutions are there to the equation $g(|x|) = 3$?

WORKING AT A*/A

(1) $f(x) = x^2 + x - 42$

(a) Sketch the graph of $y = f(x)$

(b) Solve the equation $|x|^2 + |x| - 42 = 0$

(c) Write down the minimum number of solutions to the equation $|f(x)| = a$ where a is a positive constant.

(2) (a) Sketch the graphs of $y = |\cos(x)|$ and $y = \sin(|x|)$ $-180 < x < 180$ on the same set of axes.

(b) Solve the equation $\sin(|x|) = 0.5, -180 < x < 180$.

(c) Solve the equation $\sin(|x|) = |\cos(x)|, -180 < x < 180$.

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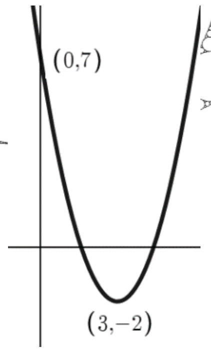
(11) Multiple Graph Transformations

WORKING AT D/E

(1) The diagram below shows part of the curve with equation $y = f(x)$. The coordinates of the minimum point and where the curve crosses the y axis is shown.

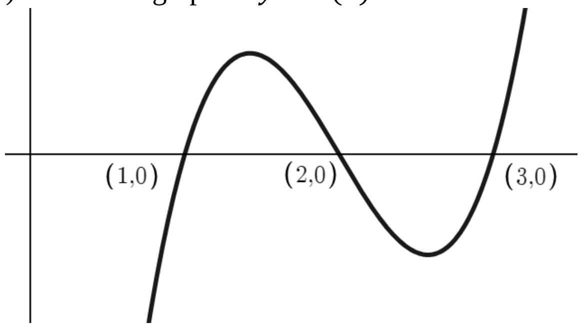


On separate diagrams, sketch each of the following:
(a) $y = 2f(x) + 1$ stating the coordinates of the minimum point and the y intercept.
(b) $y = f(2x - 5)$ stating the coordinates of the minimum point.
(c) $y = -f(x) + 3$ stating the coordinates of the maximum point and the y intercept.
(d) $y = |f(x)| - 2$ stating the coordinates of the maximum point and the y intercept.
(e) $y = -3f(0.5x)$ stating the coordinates of the maximum point and the y intercept.

(2) Describe fully the transformations that map the graphs of $y = g(x)$ to $y = 1 - g(x - 2)$.

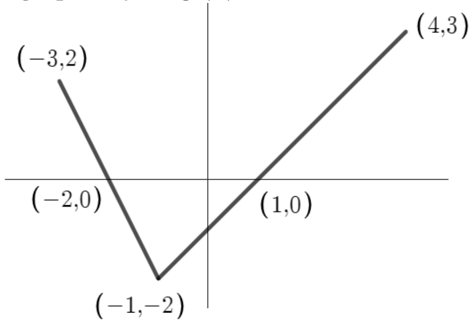
WORKING AT B/C

(1) Part of the graph of $y = h(x)$ is shown below



The coordinates shown are where $h(x) = 0$.
(a) Sketch the graphs of $y = h(2x - 1)$ showing where the graph crosses the x axis.
(b) Sketch the graph of $y = |h(x)|$ showing where the graph meets the x axis.
(c) Sketch the graph of $y = h(-x - 1)$ showing where the graph crosses the x axis.
(d) Doris wants to draw the graph of $y = ah(x)$ where a is a constant. State what will happen to the points (1,0), (2,0) and (3,0) under the transformation.

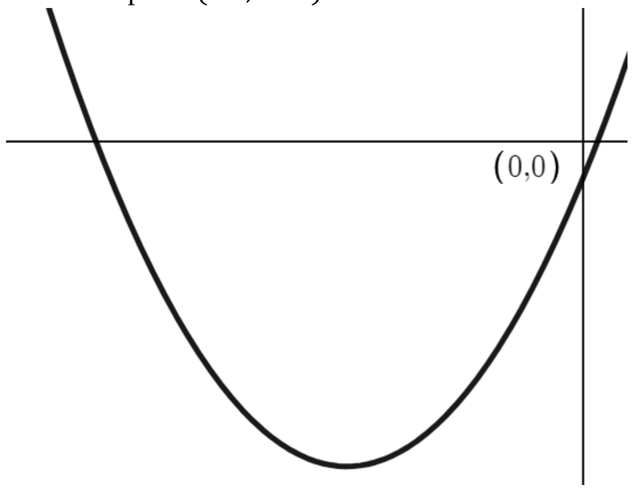
(2) The graph of $y = g(x)$ is shown below



(a) Sketch the graph of $y = 0.5g(-x) + 2$
(b) Sketch the graph of $y = -g(|x|)$
(c) Sketch the graph of $y = 1 + g(2x)$
(d) Sketch the graph of $y = |g(x + 3)|$

WORKING AT A*/A

(1) $f(x) = x^2 + bx + c,\ x \in R$ where b and c are constants.
The graph of $y = 2f(x + 3)$ is shown below with minimum point $(-4, -18)$



Find the values of b and c

(2) $t(x) = x^3 - 4x^2 + x + 6,\ x \in R$
(a) Show that $t(-1) = 0$
(b) Hence, fully factorise $t(x)$
(c) Sketch the graph of $y = t(x)$ showing where the curve crosses the coordinated axes.
(d) Sketch the graph of
$y = (4x - 2)^3 - 4(4x - 2)^2 + (4x - 2) + 6$
showing where the grapoh crosses the x axis.

(3) $f(x) = x^2 - 2x - 8,\ x \in R$
The equation $|f(x)| = a$ has 2 real solutions. Find the possible set of values of the constant a.

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(12) Solving Modulus Equations and Inequalities

WORKING AT D/E

(1) $f(x) = |x|, x \in R$
(a) Sketch the graph of $y = f(x)$
(b) Write down the range of $f(x)$
Using your answers to parts (a) and (b),
(c) Sketch the graph of $y = |x| + 3$
(d) State the range of $y = |x| + 3$
Give than $g(x) = -|x| + 3,\ x \in R$
(e) Find the range of $g(x)$
(f) Solve the equation $g(x) = -5$

(2) (a) Solve the equation $|2x + 6| = x + 4$
(b) Hence, solve $|2x + 6| \leq x + 4$

(3) By drawing two different graphs, show that there are no solutions to the equation $1 - |x| = 6$

WORKING AT B/C

(1) (a) Solve $|3x + 7| = |x - 3|$
(b) Hence, solve $|3x + 7| < |x - 3|$

(2) $f(x) = |3x| - 5$
(a) Sketch the graph of $y = f(x)$ showing where the graph crosses the coordinate axes.
(b) Solve the inequality $f(x) > 6$ giving your answers as exact fractions.
(c) Explain why $f^{-1}(x)$ doesn't exist.
(d) With the help of a sketch, show that there are no solutions to the equation $f(x) = x - 6$

(3) Sketch the graph of $y = 5 - |x + 1|$ show where the graph crosses the coordinate axes.

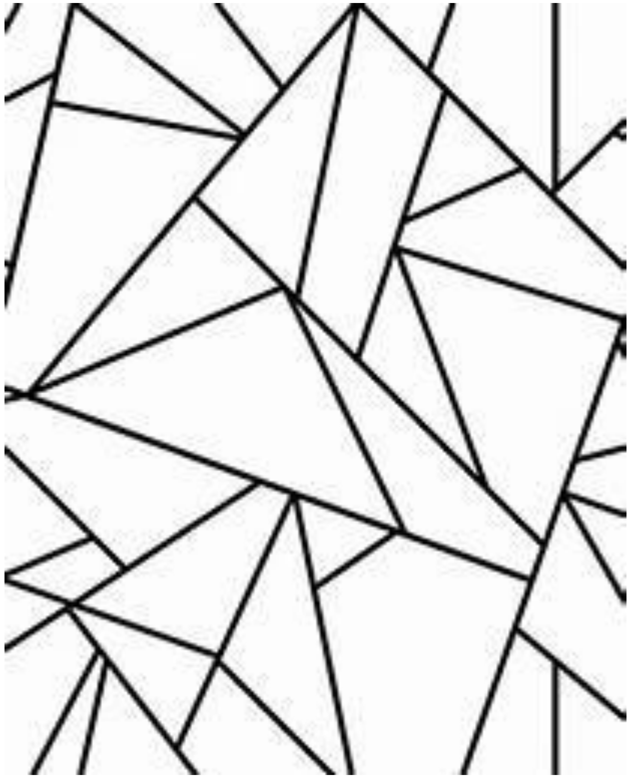
WORKING AT A*/A

(1) $f(x) = |ax - 2|, x \in R$ where a is a positive constant.
(a) Sketch the graph of $y = f(x)$ showing where the graph meets or crosses the coordinate axes. Give the coordinates in terms of a
(b) Solve $|ax - 2| \geq a$ giving your answers in terms of a
Given that there is one solution to the equation $|ax - 2| = b - x$ where b is a constant
(c) Find b in terms of a.

(2) $f(x) = |x + a| + b,\ x \in R$ where a and b are constants.
The graph of $y = |x + a| + b$ has a minimum point with coordinates $(-1,4)$ and y intercept $(0,5)$.
(a) Find the values of a and b
(b) Hence solve $|x + a| + b < 3 - 0.5x$
(c) The equation $f(x) + c = 7$ where c is a constant has one solution. Find the value of c.

(3) There are no solutions to the equation $3|\cos(x)| + 1 = a$, $0 < x < 360$, where a is a constant. Find the possible set of values of a.

Sequences and Series



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(13) Arithmetic Sequences

WORKING AT D/E

(1) The nth term of an arithmetic sequence is

$$u_n = 2n + 5$$

(a) Find the first 3 terms in the sequence.

(b) Show that the term 91 is in the sequence.

(c) Given that $u_a = 47$, find the value of a.

(2) (a) A sequence is generated by the formula

$$u_n = 24 - 3n$$

(i) How many terms in the sequence are > 0 ?
(ii) Find u_{20}

(b) A sequence is generated by the formula

$$u_n = 12 + 8n$$

(i) Find the value of a, the first term.
(ii) Write down d, the common difference.
(iii) Write down the first term that exceeds 100.

(3) Find the nth term of the sequence -10, -14, -18, -22….. in the form $u_n = pn + q$

WORKING AT B/C

(1) In an arithmetic sequence the 4th term is 18 and the 12th term is 34.

(a) Find the nth term of the sequence in the form $u_n = pn + q$, where p and q are constants.

(b) Another arithmetic sequence has nth term $u_n = 40 - 3n$. Show that there is a term in **both** sequences, stating the term.

(2) (a) Find out how many terms there are in each of the sequences below:

(i) $4, 7, 10 \ldots\ldots 238, 241$ (ii) $5, 3, 1 \ldots\ldots -121$

(b) An arithmetic sequence is $40, 36, 32 \ldots\ldots -236$. How many negative terms are there in the sequence?

(3) The first 3 terms of an arithmetic sequence are $2p - 1$, $p - 2$ and $4p + 9$.

(a) Show that $-p - 1 = 3p + 11$

(b) Hence, find the value of p

(c) Find the nth term of the formula.

(d) Write down the number of negative terms in the sequence.

WORKING AT A*/A

(1) An arithmetic sequence u_n has first term $p^2 + 1$ and second term $3p + 10$, where p is a positive constant. Given that the common difference, d, in the sequence is 5.
(a) Find the value of p
(b) Find the third term.
(c) Find the largest term less than 100.
Given instead p was a negative constant,
(d) Find an expression form the nth term of the sequence in the form $u_r = ar + b$
(e) Write down the only terms that appear in u_n that don't appear in u_r.

(2) A sequence has first 3 terms $p^2, 4p$ and $2p + 10$ where p is a constant.

Prove that the sequence is not arithmetic.

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(14) Arithmetic Series

WORKING AT D/E

(1) Find the sum of the first 80 terms in each arithmetic sequence using the formula book:
(a) First term (a) 4, common different (d) 6
(b) -3, 1, 5, 9, 13……..
(c) $u_n = 3n - 4$
(d) $u_n = 8 - n$

(2) An arithmetic series has first term 6, second term 11 and last term 611.
Use the formula book to show that the sum of all the terms in the sequence is 37637.

(3) An arithmetic series has first term -2 and second term -5. The sum of the first n terms is -5430.
(a) Using the formula book, show that

$$3n^2 + n - 10860 = 0$$

(b) Hence find the value of n

WORKING AT B/C

(1) Show that the sum of the first 40 even numbers is 1640

(2) The first term of an arithmetic series is 9 and the last is 384. Given that the sum of the terms in the series is 14541

(a) Find the number of terms in the sequence

(b) Show that the 12^{th} term in the sequence is $\frac{4782}{73}$

(3) The first 3 terms of an arithmetic series are $2p,\ p^2 - 11$ and $p + 5$ where p is a constant. Given that the sum of the first 3 terms is -6

(a) Find the value of p

(b) Find the 14^{th} term of the series

(c) Find the sum of the first 60 terms in the series.

WORKING AT A*/A

(1) The 8^{th} term of aa arithmetic series is 28 and the 14^{th} terms is 64.
(a) Find the sum of the first 100 terms in the series.
(b) Given that the sum of the first r terms doesn't exceed 3000, find the value of r.
(c) Given that the sum of the first k terms in the sequence is negative, find the greatest possible value of k.

(2) Prove that the sum of the first n terms in an arithmetic series can be given as $\frac{n}{2}(a + l)$ where a is the first term and l is the last term.

(3) Show that the difference between the sum of the first 100 even numbers and the first 100 odd numbers is 100.

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(15) Geometric Sequences

WORKING AT D/E

(1) Find the 5th and 12th term in each of the following geometric sequences:

(a) (i) 3, 5.4, 9.72…… (ii) 5^{n-1}

(b) Explain why 1.9, 6.08, 19.456, 60.3136….. is not a geometric sequence.

(2) The 5th term of a geometric sequence is 0.0512. Given that the first term is 2, Show that the common ratio $r = \pm 0.4$.

(3) Given that a geometric series with first term 2 has 7th term $\frac{2}{15625}$, find the possible values of the common ratio r.

WORKING AT B/C

(1) The 5th term of a geometric sequence is 3.1104 and the 7th term of the sequence is 4.478976
(a) Find the common ratio r, given that $r > 0$
(b) Find the first term a
(c) Find the 12th term of the sequence
(d) Find the first term in the sequence that exceeds 200.

(2) A geometric series has first three terms $p+1$, $4p$ and $12p$ where p is a constant.
(a) Write down the value of p
(b) Hence find the first 3 terms.
(c) Write down nth term for the formula in the form $c \times d^{n-1}$
(d) Find how many terms in the sequence are less than 500.

(3) A sequence has first term 10, second term 5 and so on such that it forms a geometric progression. Find the term in the sequence that is closest to 0.01

WORKING AT A*/A

(1) The first 3 terms of a geometric sequence are $k,\ 2k-11$ and $\frac{3k+1}{k}$ where k is a constant. Given that there is only one positive term in the sequence, find the value of k.

(2) A geometric sequence has first term a and common ratio r. Given that the 4th term in the sequence is 100,
(a) Explain why both a and r must be positive or both be negative.
Given that a and r are positive,
(b) Show that:

$$\log(r) = \frac{2 - \log(a)}{3}$$

(c) Given that $0 < r < 1$, find the possible set of values of a.

(3) Prove that the sequence $a, a+1, a+2$ ….. where a is a constant, is not geometric.

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(16) Geometric Series

WORKING AT D/E

(1) Using the formula book, find the sum of the first 40 terms in each geometric series:

(a) $a = 3$ and $r = 1.2$

(b) 2, 1.6, 1.28, 1.024……

(2) The sum of the first 5 terms of geometric series with common ratio 1.5 is 26.375. Use the formula book to find the first term a.

(3) 8, 6.4, 5.12…………2.62144 is a geometric series.
(a) Find the number of terms in the series.
(b) Find the sum of the terms in the geometric series

WORKING AT B/C

(1) A geometric series has 4th term 8.64 and 7th term 14.92992.
(a) Find the first term a
(b) Find the common ratio r
(c) Hence, find the sum of the first 7 terms to 3 significant figures.

(2) A geometric series has first term 2 and common ratio 1.8. Given that the sum of the first n terms of the series exceeds 25,
(a) Using the formula book, show that $1.8^n > 11$
(b) Hence, find the smallest possible value of n

(3) The first 3 terms of a geometric series are

$k,\ k + 4$ and $3k + 4$….

where k is a positive constant.

(a) Show that $k^2 - 2k - 8 = 0$

(b) Hence, find the value of k

(c) Find the sum of the first 10 terms of the series.

WORKING AT A*/A

(1) A series u_n is given by $u_n = (a \times 2^{n-1}) + 4n$ where a is a positive constant.

Given that $S_{20} = 1678560$, find the value of a

(2) Prove that the sum of the first n terms of a geometric series with first term a and ratio r is

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

(3) Prove that the sum of a geometric series with first term 4 and ratio 0.4 cannot exceed 7.

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(17) Geometric Series. The Sum to Infinity

WORKING AT D/E

(1) Circle which ones of the following series are convergent.

2, 4, 8, 16….. 10, 5, 2.5, 1.25… $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ …..

(2) Find the sum to infinity of each of the geometric series below

(a) 8, 4, 2, 1…..

(b) $a = 14,\ r = 0.1$

(3) The sum to infinity of a series with first term 12 is 40.

(a) Find the common ratio

(b) Find the 6th term.

(c) Find the sum of the first 4 terms.

WORKING AT B/C

(1) A geometric series has first term 10 and third term 6.4. Find the sum to infinity of the series,

(2) The first 3 terms of a geometric series are $p,\ 2p$ and $4p$ where p is a constant,

(a) Explain why it is not possible to find the sum to infinity for the series.

Given that p is actually 1.84,

(b) Find the sum of the first 8 terms of the series.

(c) Given that the nth term in the sequence is the first to exceed 100, find the value of n

(3) In a geometric series S_∞= 50 and the second term is 12.

(a) Show, using the formula book, that

$$50(1 - r)r = 12$$

(b) Hence, find the 2 possible values of r

WORKING AT A*/A

(1) The 5th term of a geometric series is 0.01 and the 8th term is -0.00001

Find the sum to infinity of the series,

(2) A geometric series has first 3 terms $4p,\ p - 0.8$ and $\frac{4}{25}p$ where p is a constant.

Given that $S_\infty = 20$ and all the terms in the series are positive, find the value of p.

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(18) Sigma Notation for Series

WORKING AT D/E

(1) (a) Write down what type of sequence $7+2r$ is.

(b) Hence, using the formula book, find

$$\sum_{r=1}^{45} 7+2r$$

(c) Write down what type of sequence 3×2^r is.

(d) Hence, using the formula book, find

$$\sum_{r=1}^{20} 3\times 2^r$$

(2) Evaluate

$$\sum_{r=1}^{40} 5-6r$$

(3) Evaluate

$$\sum_{r=1}^{100} 4\times 0.5^r$$

WORKING AT B/C

(1) Evaluate

$$\sum_{r=10}^{50} 5-6r$$

(2) Give that

$$\sum_{r=1}^{10} p\times 3^r = 44286$$

use the formula book to find the value of the constant p.

(3) Give that

$$\sum_{r=1}^{n} 7n-6 = 5226$$

use the formula book to find the value of n.

WORKING AT A*/A

(1) Evaluate

$$\sum_{r=10}^{20} 2\times 1.1^r + 2r$$

Giving your answer to 1 decimal place.

(2) Show that

$$\sum_{r=1}^{n+1} 4\times 2^r = 2^{n+4}-2^3$$

(3) Given that

$$\sum_{r=1}^{2} 3\times R^r = 5.13$$

where R is a constant, find

$$\sum_{r=1}^{\infty} 3\times R^r$$

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(19) Recurrence Relations and Periodic Sequences

WORKING AT D/E

(1) $u_{n+1} = 4u_n - 1,\ u_1 = 3.$

(a) Find u_2, u_3 and u_4.

(b) Explain why the sequence is not arithmetic.

(c) Find

$$\sum_{r=1}^{5} u_r$$

(2) $u_{n+1} = -u_n,\ u_1 = 4.$

(a) Find u_2, u_3 and u_4.

(b) Name what type of sequence this is.

(c) Explain why

$$\sum_{r=1}^{2000} u_r = 0$$

WORKING AT B/C

(1) $u_n = \cos(90n^o)\ , n \geq 1$

(a) Show that the order of the sequence is 4

(b) Explain why

$$\sum_{r=1}^{4n+2} u_r = -1$$

(2) $u_{n+1} = (u_n)^2 - 1,\ u_1 = p,\ p > 0$

(a) Find and expression for u_2,

Given that

$$\sum_{r=1}^{2} u_r = 19$$

(b) Find the value of p

(3) $u_n = (-1)^n,\ n \geq 1$

(a) Show that the sequence is periodic and state its period.

(b) Write down the value of

$$\sum_{r=1}^{8001} u_r$$

WORKING AT A*/A

(1) A sequence is defined for $n\ \geq 2$ by the recurrence relation

$$u_n = u_{n-1} - 3,\ u_1 = k,$$

(a) Show that the sequence is arithmetic.

Given that $u_8 = -11$

(b) Find the value of k

(c) Evaluate

$$\sum_{r=1}^{40} u_r$$

(2) $u_n = \tan(180n^o) + \cos(180n^o)\ , n \geq 1$

Explain why

(a)

$$\sum_{r=1}^{2n} u_r = 0$$

(b)

$$\sum_{r=1}^{2n+1} u_r = -1$$

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(20) Application of Series

WORKING AT D/E

(1) Doris has 1kg of chocolate. She eats half of the bar one day, then half of the remaining amount the next day and so on such that she has 500g on day one, 250g on day two, 125g on day three and so on. The amount of chocolate **remaining** can be modelled by the equation $M = 1000 \times 0.5^n$, where M is the mass in grams and n is the number of days since she started eating it.
(a) Show, using the formula, that on the 5th day of eating she will eat 31.25g of chocolate.
(b) Find the total amount of chocolate she has eaten after 8 days, giving your answer to the nearest gram.

(2) Cyril is given £10 pocket money by his parents one day to start his savings, Each week after this his pocket money increases by £2
(a) Find out how much pocket money he will get in the 18th week.
Cyril saves all of his pocket money for 20 weeks.
(b) Use the formula book to find the total amount he has saved.

WORKING AT B/C

(1) Doris is selling trees. She charges £20 for a 30cm tree which is the smallest available height. For every 10cm increase in the height of the tree she charges an additional £5 such that a 40cm tree costs £25, a 50cm tree costs £30 and so on.
(a) Explain why the cost of a tree follows an arithmetic progression.
(b) Find the cost of buying a 2.4m high tree.
Cyril wants to make a decorative garden. He decides to buy the first 20 sizes of tree Doris sells.
(c) Find the total cost of the trees Cyril buys.
(d) Cyril's father has £800 to spend on a tree. Work out the height of the largest possible tree he can buy.

(2) Doris sets up an Instagram page. On the first day 3 new people begin to follow her. Every day after that double the number of new people follow her as they did the day before such that on the 2nd day 6 new people follow, on the 3rd day 12 new people follow and so on.
The number of new (N) people following her after (t) days can be modelled by the formula

$$N = 3 \times 2^{t-1}, \qquad t \geq 1$$

(a) Use the formula to show that 24 people followed Doris on the 4th day
(b) Find out how long it will take before 1000 new people follow her each day.
(c) Find out the total number new followers she will have after 12 days.
(d) Comment on the suitability of the model.

WORKING AT A*/A

(1) Cyril is training for a marathon. He finds a circuit around his local park. On the first day he plans to run one lap. Every day after this he plans to run one more than twice as many laps as did the previous day such that on day one he run would run 1 lap, day two he would run 3 laps, day three he runs 7 laps and so on.

(a) Explain why this model is neither an arithmetic series or geometric series.

(b) Write a model using a recurrence relation for the number of laps (u) after (n) days Cyril runs.

(c) Using your answer to part (b), evaluate

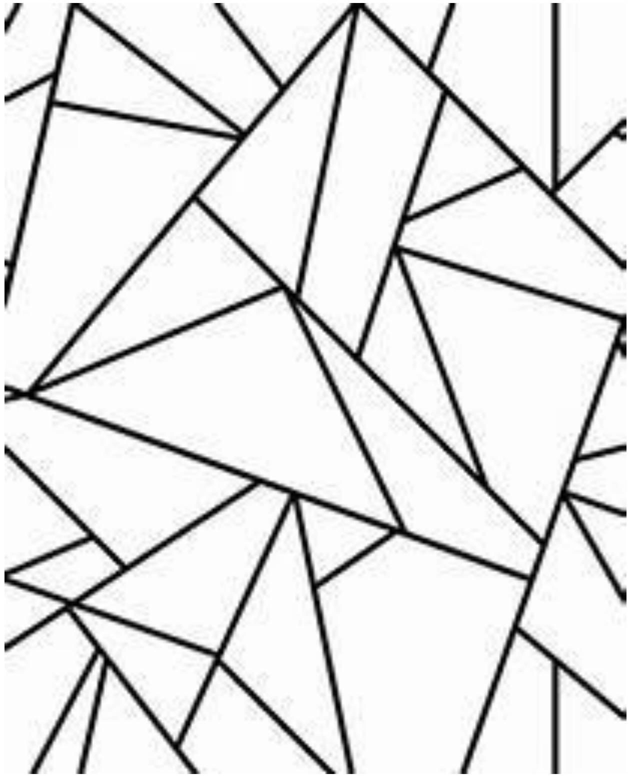
$$\sum_{r=1}^{5} u_r$$

(d) Explain what your answer to part (c) represents in the context of the question.

(e) Given that each lap was 1000m, comment on the suitability of the model.

(2) Doris is washing dishes for her parents. On day one she is paid 1p and every day after than she is paid twice as much as she was the day before such that on day two she is paid 2p, on day three she is paid 4p and so on. Doris plans to save all of the money she is paid. Find out how long it will take her to save at least £1'000'000.

Binomial Expansion



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(21) Binomial Expansion of the form $(1+x)^n$

WORKING AT D/E

(1) Using the formula book, find the first 4 terms in each of the expansions below in ascending powers of x, simplifying each term.

(a) $(1+2x)^{\frac{1}{4}},\ |x|<\frac{1}{2}$

(b) $(1-x)^{-1},\ |x|<1$

(c) $\left(1+\frac{x}{3}\right)^{-3},\ |x|<3$

(2) $f(x)=\sqrt{1+4x},\ |x|<\frac{1}{4}$

(a) Show that $f(x)$ can be written in the form $(1+4x)^n$,where n is a rational fraction.

(b) Hence, find the first 4 terms in the expansion of $f(x)$ in ascending powers of x, simplifying each term.

(c) Find the simplified value for $f(-0.1)$

(d) Using your answer to part (b), and a suitable value of x find an approximation for $\frac{\sqrt{15}}{5}$

(3) Show that the first 4 terms in the expansion of $\frac{1}{1+x}$, $|x|<1$ in ascending powers of x are:

$$1-x+x^2-x^3$$

WORKING AT B/C

(1) Using the formula book, or otherwise, find the first 4 terms in each of the expansions below in ascending powers of x, simplifying each term. State the set of values for which each expansion is valid.

(a) $\frac{1}{\sqrt{1-x}}$

(b) $\sqrt[3]{1+0.25x}$

(2) Show that the first 3 terms in the expansion of $\frac{2+x}{(1-x)^2}$, $|x|<1$ in ascending powers of x are

$$2+5x+8x^2 \dots \dots$$

(3) (a) Find the 4 terms in ascending powers of x of the expansion of $\sqrt{1-2x}$ stating the set of values of x for which the expansion ins valid.

(b) By substituting $x=0.4$ into your expansion, find an approximation of $\sqrt{5}$

(c) By considering your answer to part (a), explain how you can find a more accurate approximation to $\sqrt{5}$

WORKING AT A*/A

(1) In the expansion of $(1+px)^n,\ |x|<\frac{1}{p}$ the first 3 terms in ascending powers of x are $1-4x-4x^2$..

(a) Showing full workings, find the value of the rational constants p and n.

(b) Find the 4th term in the expansion.

(c) By choosing a suitable value of x use the expansion to find a cubic approximation to $\sqrt[3]{\left(\frac{47}{50}\right)^2}$

(2) $g(x)=\frac{\sqrt{1+3x}}{1-2x}$

Find the first 3 terms in ascending powers of x in the series expansion of $g(x)$ stating the set of values of x for which the expansion is valid.

(3) In the expansion of $\frac{2}{\sqrt[4]{1+ax}}$, $a<0$ the coefficient of the term in x^2 is 1.25. Find the coefficient of the term in x^3.

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(22) Binomial Expansion of the form $(a + bx)^n$

WORKING AT D/E

(1) (a) Show that $(4+x)^{\frac{1}{2}}$ can be written as $2\left(x+\frac{x}{4}\right)^{\frac{1}{2}}$

(b) Hence, using the formula book, find the first 3 terms in the expansion of $(4+x)^{\frac{1}{2}}$

(c) Write down the set of values of x for which the expansion ins valid.

(2) (a) Cyril wants to find the expansion of $(3-x)^{-2}$. He wants to use Pascal's Triangle to find the coefficients of each term. Explain why he can't

(b) Using the formula book, show that the first 3 terms in ascending powers of x in the expansion of $(3-x)^{-2}$ are $\frac{1}{9}+\frac{2}{27}x+\frac{1}{27}x^2$ ….

(c) Write down the set of values of x for which the expansion ins valid.

(3) Find the first 4 terms in ascending powers of x in the expansion of $(2+3x)^{-4}, |x| < \frac{2}{3}$

WORKING AT B/C

(1) Use the formula book to show that the first 3 terms in ascending powers of $x$in the expansion of $\frac{1+x}{\sqrt{9-x}},\ |x| < 9$ are $\frac{1}{3}+\frac{19}{54}x+\frac{13}{648}x^2$ …..

(2) (a) Find the first 3 terms in the expansion of $\sqrt{2+x},\ |x| < 2$ in ascending powers of x simplifying each coefficient.

(b) Use your answer to part (a) with a suitable value of x to find an approximation to value of $\frac{\sqrt{201}}{10}$

(3) Find the first 2 terms in the series expansion of $\frac{5-x}{(2+x)^2}$ stating the set of values of x for which the expansion ins valid.

WORKING AT A*/A

(1) $f(x) = (2+bx)^c$
Given that the first two terms in the binomial expansion of $f(x)$ are $\frac{1}{4}-\frac{3}{4}x$….

(a) Write down the value of c

(b) Find the value of b.

(c) Find the 3rd term in the expansion

(d) Find the set of values of x for which the expansion ins valid.

(e) Without any further expansions, find the first 3 terms in the expansion of $(2-bx)^c$

(2) (a) Find the first 4 terms in ascending powers of x in the expansion of $(x+8)^{\frac{1}{3}}$ simplifying each term.

(b) Find the set of values of x for which the expansion ins valid.

(c) Use your answer to part (b) to find a cubic approximation for $\sqrt[3]{9}$, showing all your workings.

(d) Find the percentage error in your approximation.

(e) Explain how this approximation could be improved.

(3) (a) $h(x) = \frac{1}{1+x}+\frac{1}{1-x}$, $|x| < 1$
Explain why there are no odd powers of x in the series expansion of $h(x)$.

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(23) Binomial Expansions Using Partial Fractions

WORKING AT D/E

(1) (a) Express $\frac{5+7x}{(1+x)(1+2x)}$ in partial fractions.

(b) Hence, using the formula book, show that the first 3 terms in ascending powers of x in expansion of $\frac{5+7x}{(1+x)(1+2x)}$ are

$$5-8x+14x^2 \ldots.$$

(c) Explain why $|x|<\frac{1}{2}$ instead of $|x|<1$ for the series expansion to be valid.

WORKING AT B/C

(1) (a) Express $\frac{13+7x}{(1-x)(3+x)}$ in the form $\frac{A}{(1-x)}+\frac{B}{(3+x)}$.

(b) Hence, using the formula book, find the first 4 terms in the expansion of $\frac{13+7x}{(1-x)(3+x)}$ in ascending powers of x, simplifying each term.

(c) State the set of values of x for which the expansion ins valid.

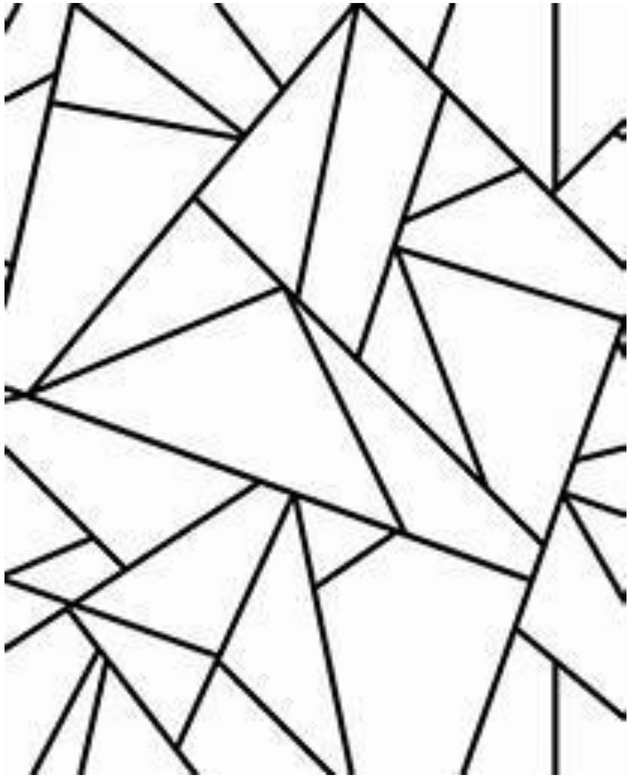
WORKING AT A*/A

(1) $h(x)=\frac{-x(x+8)}{(1-x)(2+x)^2}$, $|x|<1$

(a) Express $h(x)$ in partial fractions.

(b) Hence, find the first 3 non-zero terms in the binomial expansion of $h(x)$ simplifying each term.

Radians



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(24) Using Radians as a Measurement of Angles

WORKING AT D/E

(1) Without a calculator, convert each of the following to radians, giving your answers as multiples of π:

(a) 30^0 (b) 60^0 (c) 45^0 (d) 90^0

(e) 120^0 (f) 0^0 (g) 360^0 (h) 180^0

(2) Without a calculator, convert each of the following to degrees:

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) π (d) $\frac{\pi}{2}$ (e) 4π

(3) (a) Use a calculator to convert each of the following to degrees. Give answers to 1dp.
(i) 1.2^c (ii) 0.87^c (iii) 5.36^c

(b) Use a calculator to convert each of the following to radians giving answers to 3SF.
(i) 37^o (ii) 254^o (iii) 112^o

WORKING AT B/C

(1) Without a calculator, convert each of the following to radians, giving your answers as multiples of π:

(a) 240^0 (b) 300^0 (c) 135^0 (d) 15^0

(e) -30^0 (f) -45^0 (g) 210^0 (h) -90^0

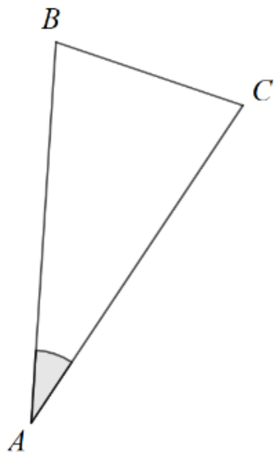
(2) Without a calculator, convert each of the following to degrees:

(a) $\frac{-2\pi}{3}$ (b) $-\frac{5\pi}{4}$ (c) 3π

(d) $-\frac{\pi}{6}$ (e) -2π (f) 8π

(3) ΔABC is shown below. $AB = \sqrt{3}$, $AC = 2$ and $< BAC = \frac{\pi}{6}$

Without using a calculator, **show that** the area of ΔABC is $\frac{\sqrt{3}}{2}$ units.

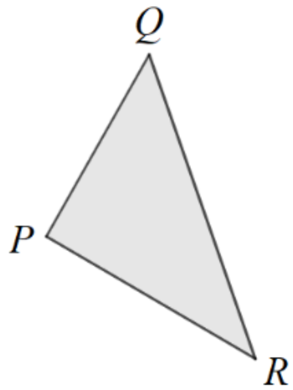


WORKING AT A*/A

(1) (a) Sketch the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$, $0 \le x \le 2\pi$ show where the curve meets or crosses the coordinate axes. Write down the coordinates of any maximum or minimum points.
(b) The graph of $y = pcos(\theta - q)$, $0 \le \theta \le 2\pi$ where p is a positive constant and $0 < q < \frac{\pi}{2}$ crosses the θ axis at $\left(\frac{5\pi}{6}, 0\right)$ and $\left(\frac{11\pi}{6}, 0\right)$.
The graph has a maximum point at $(r, 8)$.
Find the values of p, q and r giving q and r in terms of π

(2) ΔPQR is shown below. $PR = \sqrt{3}$, $QR = 2$ and $< PRQ = \frac{\pi}{6}$

Without using a calculator, find the length of PR in its simplest form.



(3) Show, without using a calculator, that $\left(\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)\right)^4$ is an integer.

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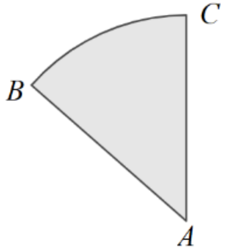
(25) Arc Lengths (Radians)

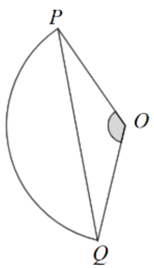
WORKING AT D/E

(1) The diagram below shows sector ABC. The angle at $A = \frac{\pi}{6}$ and $BC = 12$



(a) Without a calculator, show that the arc $BC = 2\pi$.
(b) Hence, find the perimeter of the sector ABC in the from $A + B\pi$.
(c) Given instead the angle at $A = 1.05^c$, without using a calculator, explain what impact that will have on your answer to part (a).

(2) The diagram below shows the sector OPQ. The $< POQ = \frac{2\pi}{3}$, $OP = 8$ and PQ is a straight line.

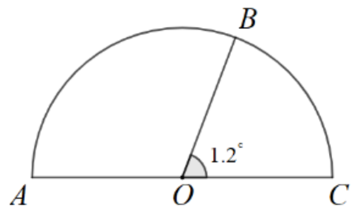


Show that the perimeter of the segment created by the line $PQ = \frac{16\pi}{3} + 8\sqrt{3}$

WORKING AT B/C

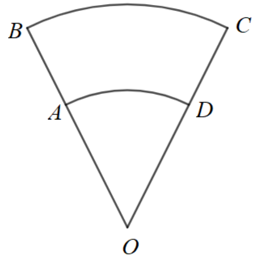
(1) (a) A sector has centre O and arc AB of length $\frac{10\pi}{3}$. Given that $AO = 4$, find $< AOB$ as a multiple of π.
(b) Write down the perimeter of the sector in exact form.
(c) A straight line AB is drawn to create a segment within the sector. Find the perimeter of the segment to 3 significant figures.

(2) The diagram below shows a semicircle centre O. $OA = OB = OC$ and $< BOC = 1.2^c$



Given that the arc length $AC = 10\pi$, show that the perimeter of the sector AOB is 39.4 to 3S.F.

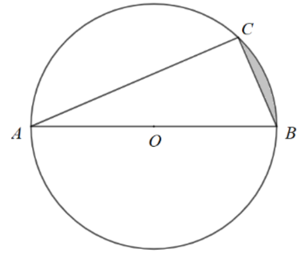
(3) The diagram below shows a sector centre O.



BC and AD are arcs of the sectors OBC and OAD respectively . The length $OB = 9, OD = 5$ and $< AOD = \frac{\pi}{4}$. Find the perimeter of shape $BACD$ in the form $P + Q\pi$.

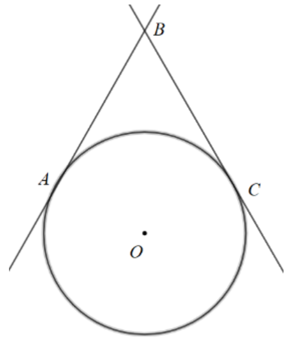
WORKING AT A*/A

(1) The diagram below shows a circle centre O and diameter $AB = 10cm$. The point C lies on the circumference of the circle. The straight line BC creates a shaded segment as shown below.



Given that $< CAB = 0.5^c$, find the perimeter of the shaded segment to 3S.F

(2) The diagram below shows a circle with centre O and radius 1. The points A and C lie on the circumference of the circle and AB and CB are tangents to the circle.



Given than $< ABC = \theta$
(a) Show that the length of the minor arc $AC = \pi - \theta$
(b) Find a simplified expression for the length of the major arc AC.
(c) A straight line AC is drawn. Shown that the area of the triangle $OAC = \frac{\sin\vartheta}{2}$

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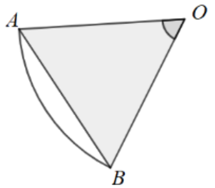
(26) Areas of Sectors and Segments (Radians)

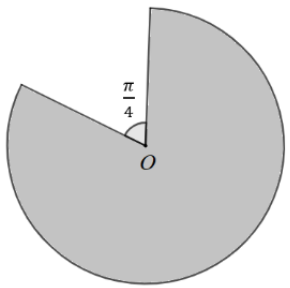
WORKING AT D/E

(1) The diagram below shows a sector centre O with radius $6cm$ and $< AOB = \frac{\pi}{3}$. A straight line AB is drawn. AB is also an arc of the sector.



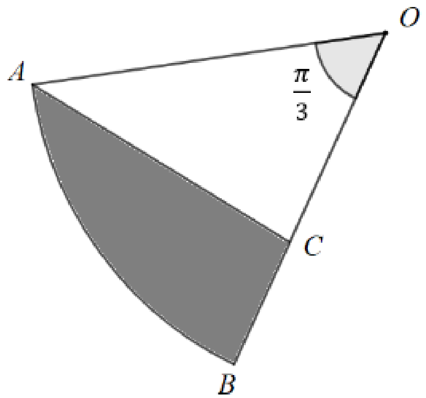
(a) Find the area of the entire sector in the form $a\pi$.
(b) Show that the area of the shaded triangle $AOB = 9\sqrt{3}\ cm^2$
(c) Hence, find the exact area of the unshaded segment shown on the diagram.

(2) The diagram below shows a major segment, centre O with radius $4cm$. The angle shown is $\frac{\pi}{4}$.
Find the **exact value** of the shaded area.



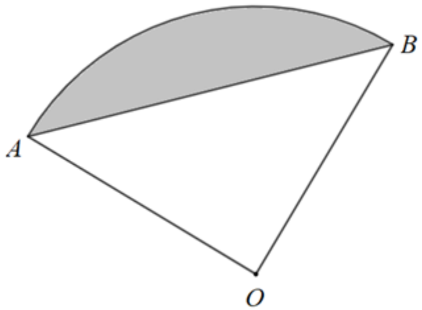
WORKING AT B/C

(1) The diagram below shows sector with radius 8 and centre O and $< AOB = \frac{\pi}{3}$



AB is a minor arc of the sector and the lines AC and OB are perpendicular.
Show that the dark shaded area is $\frac{32\pi}{3} - 8\sqrt{3}$.

(2) The diagram below shows a sector with centre O.



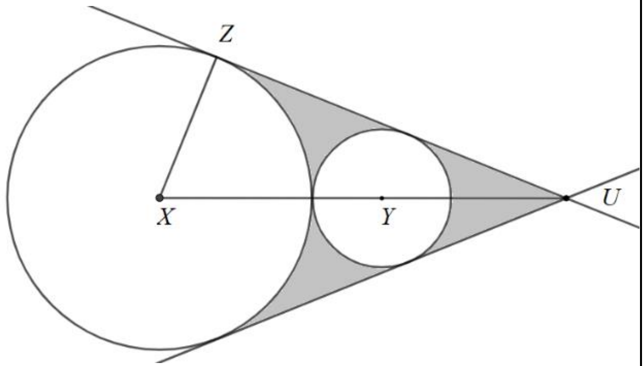
Given that the $< AOB = 1.4$ radians and the minor arc AB has length $7cm$, find the area of the shaded segment to 1 decimal place.

(3) A minor sector has radius 4 and area 8. Find the perimeter of the shape.

WORKING AT A*/A

(1) In ΔABC, $< BAC = \frac{\pi}{3}$, $BC = 13cm$ and $AB = 10\sqrt{2}cm$.
(a) Find the **least possible** size of the area of ΔABC.
The entirety of ΔABC lies inside a circle where AB is a diameter of the circle.
(b) Explain why the point C doesn't lie on the circle.
(c) What proportion of the circle does the triangle occupy?

(2) The diagram below shows two touching circles with centres X and Y. The circles touch a shared tangents that meet at the point U. The line XZ is a radius of the larger circle



Given $< ZUX = \frac{\pi}{6}$,$XZ = \sqrt{3}$ and the radius of the smaller circle is r, show that the total shaded area can be written as $3\sqrt{3} - \pi(1 + r^2)$.

(3) The area of a quarter circle is $(\pi - 2\ )cm^2$ and radius $x^{0.5}$. Find the value of x in the form $p + \frac{q}{\pi}$

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(27) Solving Trigonometric Equations (Using Radians)

WORKING AT D/E

(1) (a) Find the **two** solutions to the equation $2\sin(x) = 1$ for $0 \le x \le 2\pi$. Give your answers as multiples of π.
(b) Find the **two** solutions to the equation $\cos(x) = \frac{\sqrt{2}}{2}$ for $0 \le x \le 2\pi$. Give your answers as multiples of π.
(c) Find the **two** solutions to the equation $\tan(x) = \frac{1}{\sqrt{3}}$ for $0 \le x \le 2\pi$. Give your answers as multiples of π.

(2) One of the two solutions to the equation $\sin(x) = 0.4$ in the interval $-\pi \le x \le \pi$ is 0.412 correct to 3 significant figures. Find the other solution to 3 significant figures.

(3) (a) Find the 4 solutions to the equation $\tan(2x) = 1$ for $0 \le x \le 2\pi$ giving your answers in exact form and in radians.
(b) Find the 2 solutions in exact form for the equation $\cos\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$ for $0 \le x \le 2\pi$.
(c) Solve $\sin(3x) = 0.5,\ 0 \le x \le \pi$ giving your answers in terms of π.

WORKING AT B/C

(1) (a) Show that the equation

$$2\sin^2(x) - 3\cos(x) = 0$$

can be written as $(2\cos(x) - 1)(\cos(x) + 2) = 0$
(b) Hence, solve the equation
$2\sin^2(x) - 3\cos(x) = 0,\ \ 0 \le x \le 2\pi$

(2) (a) Show that the equation $\sqrt{3}\sin(x) = \cos(x)$ can be written in the form $\tan(x) = k$.

(b) Hence, solve the equation
$\sqrt{3}\sin(2\theta) = \cos(2\theta),\ -\pi < \theta < \pi$ giving your answers as multiples of π.

(3) Show that there are only 3 solutions to the equation $3\sin(x) = 2\sin(x)\cos(x)$ in the interval $0 \le x \le 2\pi$

WORKING AT A*/A

(1) Solve the equation
$\sin^2\left(3x - \frac{\pi}{6}\right) = \cos^2\left(3x - \frac{\pi}{6}\right),\ 0 \le x \le \pi$
giving your answers as multiples of π.

(2) Solve the equation $6\tan^2\theta = 2 - 4\tan\theta$, $-\pi < \theta < \pi$ giving your answers in radians. Give any non-exact answers to 3 significant figures.

(3) Solve the equation

$$4\cos^2(x) + 5\sin(x) - 5 = 0,\ \ 0 \le x \le 2\pi$$

Give your answers in radians. Give any non-exact values to 3 S.F

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(28) Small Angle Approximations in Trig

WORKING AT D/E

(1) Given that θ is small, use the formula book to show that $\frac{\sin(4\theta)}{\tan(8\theta)} \approx 0.5$

(2) Given that θ is small, use the formula book to find an approximation for $\frac{1-\cos(\vartheta)}{\sin(\theta)}$

(3) When θ is small, use the formula book to simplify $\frac{\sin(6\theta)}{2\theta}$

WORKING AT B/C

(1) Show that, when θ is small, $\frac{\cos(2\theta)}{\theta\sin(\theta)} \approx \frac{1-2\theta^2}{\theta^2}$

(2) (a) Use your calculator to find the value of $\cos(0.1^c)$ giving your answer to 5dp.

(b) Use the small angle approximation to show that $\cos(0.1^c) \approx 0.995$

(c) Find the percentage error for the approximation.

(3) Given that θ is small, simplify

$$\frac{\theta^2 + \cos\theta - 1}{2\sin\theta}$$

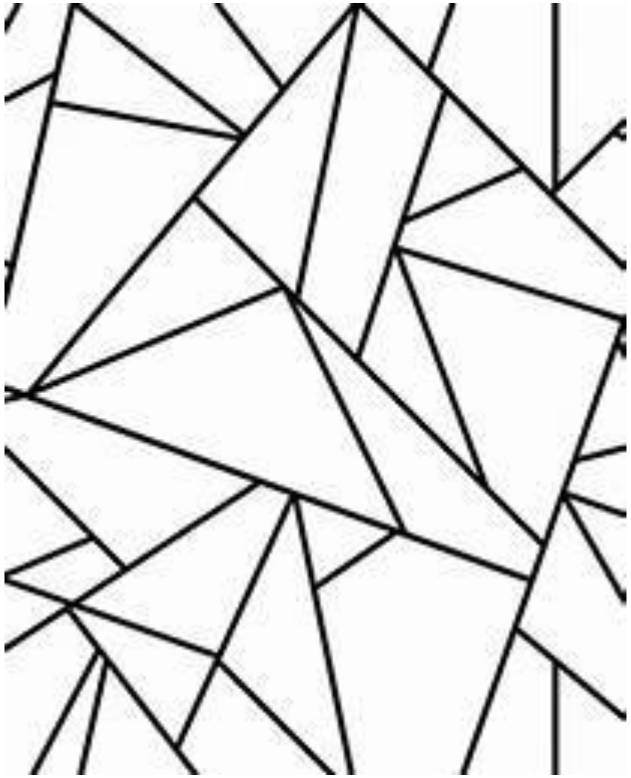
WORKING AT A*/A

(1) (a) Given that θ is small, show that

$$\frac{4 - 4\cos(2\theta) + \vartheta}{\sin(\theta)} = 8\theta + 1$$

(b) Hence, find an approximation the value of $\frac{4-4\cos(2\theta)+\vartheta}{\sin(\theta)}$ when θ is small.

Trig 1



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(29) Secant, Cosecant and Cotangent Ratios in Trig

WORKING AT D/E

(1) Complete the following sentences:

(a) If $\sin x = \frac{1}{2}$, then $\operatorname{cosec} x =$___________

(b) If $\cos x = \frac{1}{\sqrt{2}}$, then $\sec x =$___________

(c) If $\tan x = \sqrt{3}$, then $\cot x =$___________

(d) If $\sin x = -0.1$, then $\operatorname{cosec} x =$___________

(2) Without a calculator find the value of $\operatorname{cosec}(60^0)$ in the form $p\sqrt{3}$ where p is a rational fraction.

(3) Without a calculator, find the value of $\cot(-45^o)$

WORKING AT B/C

(1) Given that $3\sin x = -4\cos x$,

(a) Find the value of $\tan x$

(b) Hence write down the value of $\cot x$

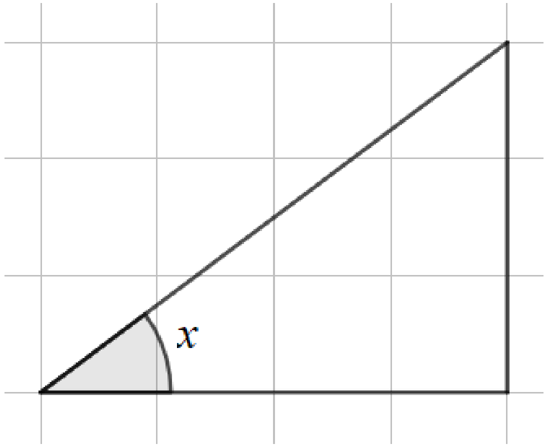
(c) Explain why x cannot be an acute angle.

(2) Without a calculator, find the value of $\frac{\sin\frac{\pi}{3}}{\cot\frac{\pi}{3}}$

(3) Given that $\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta}$, explain why $\operatorname{cosec} 180^o$ is undefined.

WORKING AT A*/A

(1) A right-angle triangle is shown on a grid below



(a) Write down the value of $\sec x$

(b) Write down the value of $\cot x$

(c) Write down the value of $\operatorname{cosec} x$

(d) Verify that $\frac{\cos x}{\sin x} \equiv \cot x$

(2) Simplify the expression $\sec(2\pi - x)$

(3) Find all the values of x for $0 \le x \le 2\pi$ where $\cot x$ is undefined giving a justification for your answers.

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(30) Sketching the Graphs of $\sec x$, $\operatorname{cosec} x$ and $\cot x$

WORKING AT D/E

(1) By first drawing the graph of $y = \sin x$, $0 \leq x \leq 360$, draw the graph of $y = \operatorname{cosec} x$ including the 3 vertical asymptotes. Write down the coordinates of the one minimum point and one maximum point.

(2) By first drawing the graph of $y = \cos x$, $0 \leq x \leq 360$, sketch the graph of $y = \sec x$ including the 2 vertical asymptotes. Write down the coordinates of the one minimum point and two maximum points.

(3) (a) Write down an expression for $\cot x$ in terms of $\tan x$.

(b) Write down when $\tan x = 0$ for $0 \leq x \leq 360$

(c) Sketch the graph of $y = \cot x$, $0 \leq x \leq 360$ showing where the graph crosses the x axis and writing down the equations of the vertical asymptotes.

WORKING AT B/C

(1) Sketch the graph of $y = \frac{2}{\sin x}$, $0 \leq x \leq 360$. Write down the coordinates of any turning points and the equations of any asymptotes.

(2) (a) Sketch the graph of $y = 2 + \sec x$, $0 \leq x \leq 360$.
(a) When $\sec x = -2$, what is the value of $\cos x$?
(b) Hence, find where the graph of $y = 2 + \sec x$, $0 \leq x \leq 360$ crosses the x axis.

(3) The graph of $y = 3\operatorname{cosec}(x - 30)$ $0 \leq x \leq 360$ has a minimum point with coordinates (p, q). Write down the values of p and q.

WORKING AT A*/A

(1) The graphs $y = p\sec x$, $p > 0$ and $y = q$ where p and q are constants, don't intersect. Find the possible set of values of q in terms of p.

(2) (a) $f(x) = 1 - \frac{a}{\operatorname{cosec} x}$, $a > 1$

(a) Sketch the graph of $y = 1 - \frac{a}{\operatorname{cosec} x}$, for $0 \leq x \leq 360$ including asymptotes.

(b) Explain why there are no roots to the equation $f(x) = 0$

(3) (a) Sketch the graphs of $y = \sec(x)$ and $y = \cot(x)$ for $-\pi \leq x \leq \pi$ on the same set of axes.

(b) Write down the number of points of intersection of the graphs of $y = \sec(x)$ and $y = \cot(x)$ for $-\pi \leq x \leq \pi$

(c) Find the coordinates any points of intersection giving any answers to 3S.F

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(31) Equations and Identities using $\sec x$, $\operatorname{cosec} x$ & $\cot x$

WORKING AT D/E

(1) Write each of the following in terms of $\sec\theta$.

(a) $\frac{1}{\cos^2\theta}$ (b) $\frac{4}{\cos 3\theta}$

Write each of the following in terms of $\operatorname{cosec}\theta$.

(c) $\frac{1}{\sin^2\theta}$ (d) $\frac{5}{\sin 2\theta}$

Write each of the following in terms of $\cot\theta$.

(e) $\frac{3}{\tan^2\theta}$ (f) $\frac{\cos 4\theta}{\sin 4\theta}$

(2) (a) Simplify $\cos x \operatorname{cosesec} x$

(b) Hence, find the 2 solutions for
$\cos x \operatorname{cosesec} x = 1, \quad 0 < x < 360$

(3) Show that there are 4 solutions to the equation
$\sec^2 x = 4, \quad 0 < x < 360$

WORKING AT B/C

(1) (a) Show that $\operatorname{cosec} 2\theta \tan 2\theta \equiv \sec 2\theta$

(b) Hence, solve the equation

$$\operatorname{cosec} 2\theta \tan 2\theta = \sqrt{2}, \quad 0 \le \theta \le 2\pi,$$

giving your answers as multiples of π

(2) Show that

$$\frac{(\cos x + \sin x)^2}{\cos x} \equiv \sec x + 2\sin x$$

(3) (a) Show that, if $\cot^2\theta - 2\cot\theta - 8 = 0$, then $\tan\theta = 0.25$ or $\tan\theta = -0.5$.

(b) Hence, solve the equation

$$\cot^2\theta - 2\cot\theta - 8 = 0\ , \quad 0 < \theta < 360$$

Give your answers to 3SF.

WORKING AT A*/A

(1) Show that there are no solutions to the equation
$20\operatorname{cosec}^2\theta + 7\operatorname{cosec}\theta = 6, \quad 0 \le \theta \le 2\pi$

(2) Given that $\cot p = \frac{4}{3}$ where p is a reflex angle measure in radians, find the value of:

(a) $\cos^2 p$

(b) $\operatorname{cosec}^2 p$

(c) $\sin p$

(d) $\sin\left(\frac{\pi}{2} - p\right)$

(3) (a) Show that

$$(\cot x + \tan x)^2 \equiv \operatorname{cosec}^2 x \sec^2 x$$

(b) Hence, or otherwise, show that there are no solutions to the equation $(\cot x + \tan x)^2 = 0$, $0 \le x \le 360$

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(32) Reciprocal Trigonometric Identities

WORKING AT D/E

(1) Using the identity $\sin^2\theta + \cos^2\theta \equiv 1$, show that:

(a) $\tan^2\theta + 1 \equiv \sec^2\theta$

(b) $1 + \cot^2\theta \equiv \mathrm{cosec}^2\,\theta$

(2) Given that $\sec x = 4$, show that $\tan x = \pm\sqrt{15}$

(3) Using the identities in question (1), prove each of the following identities:

(a) $(\cot x + 1)^2 - \mathrm{cosec}^2\,x \equiv 2\cot x$

(b) $\tan^2\theta - \cot^2\theta \equiv \sec^2\theta - \mathrm{cosec}^2\,\theta$

WORKING AT B/C

(1) (a) Show that the equation

$$3\cot^2\theta - 5\,\mathrm{cosec}\,\theta + 1 = 0$$

can be written as

$$(3\,\mathrm{cosec}\,\theta + 1)(\mathrm{cosec}\,\theta - 2) = 0$$

(b) Hence, solve the equation

$$3\cot^2\theta - 5\,\mathrm{cosec}\,\theta + 1 = 0, \quad 0 < \theta < 360$$

(2) Given that $\cot A = \frac{5}{12}$, $0 < A < 90$, find the value of:

(a) $\tan A$

(b) $\mathrm{cosec}\,A$

(c) If $90 \leq A < 180$ instead, how would this change the answers to part (a) and (b) in the question?

(3) Solve the equation $3\sec x = 4\,\mathrm{cosec}\,x$, $0 \leq x \leq 2\pi$, giving your answers in radians to 3SF.

WORKING AT A*/A

(1) Find the exact solutions to the equation

$$\sqrt{3}\sec^2\theta - 2\tan\theta - 2\sqrt{3} = 0, \quad 0 \leq \theta \leq 2\pi,$$

(2) Prove the identity

$$\frac{\sec^4\theta - \tan^4\theta}{\tan^2\theta} \equiv \sec^2\theta\cot^2\theta + 1$$

(3) Given that:

$$A = p\,\mathrm{cosec}\,x$$

$$B = q\cot x$$

(a) Show that $(Aq)^2 - (Bp)^2 = (pq)^2$, where p and q are non-zero constants.

(b) Given that x is an acute angle, show that $\cos x \equiv \frac{\sqrt{A^2-p^2}}{A}$

(c) Given further that $A = 2$ and $p = 1$, find the value of x giving your exact answer in radians.

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(33) Inverse Trig Functions $\arcsin x$, $\arccos x$ and $\arctan x$

WORKING AT D/E

(1) Find the value of each in radians in terms of π:

(a) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$ (b) $\arccos(0.5)$ (c) $\arctan(\sqrt{3})$

(2) Write down the value of each:

(a) $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$ (b) $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right)$

(3) Find, without a calculator, the value of each:

(a) $\cos(\arcsin(1))$ (b) $\tan\left(\arccos\left(\frac{1}{\sqrt{2}}\right)\right)$

WORKING AT B/C

(1) (a) Sketch the graph of $y = \arccos x$ stating the domain and range.

(b) Sketch the graph of $y = \arcsin x$ stating the domain and range.

(c) Sketch the graph of $y = \arctan x$ stating the domain, range and the equations of any asymptotes.

(2) Given that $\arcsin a = x,\ 0 < x < \frac{\pi}{2}$, show that:

(a) $\cos x = \sqrt{1 - a^2}$

(a) $\tan x = \frac{a}{\sqrt{1-a^2}}$

WORKING AT A*/A

(1) Given that $\arccos k = x,\ 0 < x < \frac{\pi}{2}$, find an expression for;

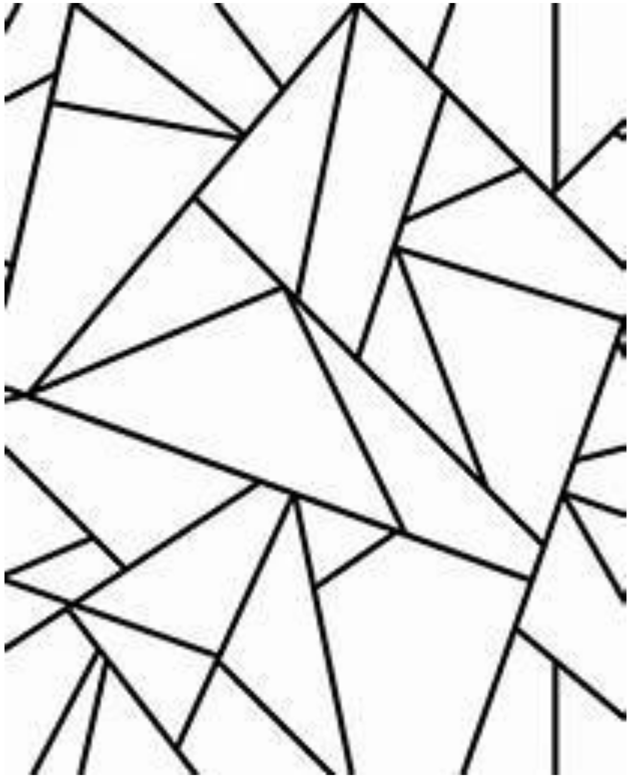
(a) $\sin x$

(b) $\tan x$

(c) $\cos(\arccos k)$

(2) Given that $\arctan\left(\frac{\pi}{2} - k\right) = y,\ 0 < y < \frac{\pi}{2}$, find an expression for $\sin y$.

Trig 2



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(34) Addition Formulae $\sin(A \pm B)$ & $\cos(A \pm B)$

WORKING AT D/E

(1) Using the formula book, prove each of the following identities:

(a) $\sin(90^o - x) \equiv \cos x$

(b) $\cos(90^o - x) \equiv \sin x$

(c) $\sin(30^o + x) \equiv \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$

(2) Cyril is trying to find the expansion for $\tan(45^o + x)$

He writes:

$$\tan(45^o + x) = \tan 45^o + \tan x$$
$$= 1 + \tan x$$

(a) Explain what he has done wrong.

(b) Use the formula book to find the correct expansion for $\tan(45^o + x)$

(3) Show that $\cos(\pi + x)$ can be written as $-\cos x$ by using the addition formulae in the formula book.

WORKING AT B/C

(1) Write $\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x$ in the form:

(a) $\sin(A + B)$

(b) $\cos(A - B)$

(2) (a) Write down the expansion of $\sin(A + B)$

(b) Write down the expansion of $\cos(A + B)$

(c) Using your answers to part (a) and (b), show that that $\tan(A + B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(3) Given that $\cos(A - B) \equiv -\sin B$, where A is a reflex angle, find the value of A in radians.

WORKING AT A*/A

(1) Given that $p \sin\left(\frac{\pi}{2} + x\right) = q \cos\left(\frac{\pi}{2} + x\right)$, write an expression for $\cot x$ in terms of the constants p and q.

(2) Given that $4\sin(x - y) = \cos(x + y)$, show that $\tan x = \frac{4\tan y + 1}{4 + \tan y}$

(3) Write $-\sin A$ in the form $\cos(A + B)$ where $0 < B \leq \pi$.

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(35) Applying the Addition Formulae in Trig

WORKING AT D/E

(1) Using the formula book with $A = 45^o$ and $B = 30^o$ show that the exact value of $\sin(75^o)$ is $\frac{\sqrt{2}+\sqrt{6}}{4}$.

(2) Use the formula book to write $\cos 60 \cos 45 + \sin 60 \sin 45$ in the form $\cos P$ where P is an acute angle.

(3) (a) Using the formula book, show that

$$\tan 15^0 = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

(b) Hence, without a calculator, show that

$$\tan 15^o = 2 - \sqrt{3}$$

WORKING AT B/C

(1) Using the expansion of $\cos(A + B)$ show that

$$\cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$$

(2) Write $\frac{1+\sqrt{3}}{1-\sqrt{3}}$ in the form $\tan P$, $\frac{\pi}{2} < P < \pi$

(3) Given that $\cos A = \frac{1}{3}$ and $\cos B = \frac{1}{6}$ where A and B are both acute angles, show that:

(a) $\sin A = \frac{2\sqrt{2}}{3}$

(b) $\sin B = \frac{\sqrt{35}}{6}$

(c) $\cos(A + B) = \frac{1-2\sqrt{70}}{18}$

(d) $\sin(A + B) = \frac{2\sqrt{2}+\sqrt{35}}{18}$

(e) Using a calculator, show that $\tan(A + B) = -0.556$ correct to 3 significant figures.

WORKING AT A*/A

(1) Given that $\tan A = p$ and $\tan B = q$ where A is an acute angle and B is a reflex angle, show that

$$\cos(A + B) = \frac{1-pq}{\left(\sqrt{1+p^2}\right)\left(\sqrt{1+q^2}\right)}$$

(2) Given that $\sin A = 0.8$ and $\cos B = 0.6$, where A is an obtuse angle and B is a reflex angle, show that $\cot(A + B) = \frac{7}{24}$

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(36) Double Angle Formula $\cos 2A$, $\sin 2A$ and $\tan 2A$

WORKING AT D/E

(1) Use the formula book to show that

$$\sin(2A) \equiv 2\sin A\cos A$$

(2) Use the formula book to show that $\cos(2\theta)$ can be written as:

(a) $\cos^2\theta - \sin^2\theta$

(b) $2\cos^2\theta - 1$

(c) $1 - 2\sin^2\theta$

(3) Prove that $\cos(2\theta) + \cos^2\theta + \sin^2\theta \equiv 2\cos^2\theta$

WORKING AT B/C

(1) Given that $x = \cos 2\theta$ $y = \sin\theta$, show that $2y^2 + x = 1$

(2) (a) Prove that

$$\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x} \equiv \tan 2x$$

(b) Hence, solve $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x} = 1$ for $0 < x < \pi$ giving your answers in terms of π.

(3) Solve the equation

$$\cos 2x + \cos x = 0, \quad 0 \leq x \leq 360$$

WORKING AT A*/A

(1) (a) Prove that

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

(b) Hence, solve the equation

$$\frac{3\tan 2y - \tan^3 2y}{1 - 3\tan^2 2y} = \sqrt{3}, \ 0 < y < 90$$

(2) Given that A is an acute angle and $\cos 2A = \frac{1}{3}$ find an expression for:

(a) $\sin A$

(b) $\cos A$

(c) $\sin 2A$

(d) $\cot 2A$

(3) (a) Solve the equation

$$\sin 2y = \tan y, \quad 0 \leq y \leq 360$$

(a) Solve the equation

$$2\cos^2\frac{\theta}{2} = \sin\theta + 1 \qquad 0 < \theta < 2\pi$$

giving your answers in exact form

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A LEVEL MATHS

(37) Solving Trigonometric Equations

WORKING AT D/E

(1) (a) Use the formula book to show that $\sin 2x \equiv 2\sin x\cos x$

(b) Hence, show that $\sin 2x = \sin x$ can be written as $\sin x(2\cos x - 1) = 0$

(c) Using your answer to part (b) find the 5 solutions to the equation $\sin 2x = \sin x$, $0 \le x \le 360^o$

(2) (a) Using the formula book to show that

$$\cos(x-30) \equiv \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x$$

(b) Using your answer to part (a), solve the equation

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = 1, \quad 0 \le x \le 360$$

(3) Using the formula book or otherwise, solve the equation: $\frac{2\tan x}{1-\tan^2 x} = 1$, $0 \le x \le 360$

WORKING AT B/C

(1) (a) Use the formula book with $A = \frac{1}{2}x$ and $B = \frac{1}{2}x$ so show that $\cos x = \cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$

(b) Hence, or otherwise, solve the equation

$$\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x = 0.1, \quad 0 \le x \le 360^o$$

Give each answer to 1 decimal place.

(2) (a) Write $4\sin x\cos x$ in the form $p\sin qx$

(b) Hence, solve the equation $4\sin x\cos x = 1$, $0 \le x \le 2\pi$, giving your answers as multiples of π

(3) (a) Show that

$$6\sin x\cos x = 2(\cos^2 x - \sin^2 x)$$

can be written as $\tan 2x = \frac{2}{3}$

(b) Hence, solve the equation

$$6\sin x\cos x = 2(\cos^2 x - \sin^2 x), \quad 0 < x < 360$$

giving each answer to 3SF.

WORKING AT A*/A

(1) (a) Show that $2\cos x - 2\cos^3 x \equiv \sin x\sin 2x$

(b) Hence, or otherwise, find the exact solutions to the equation $2\cos x = 2\cos^3 x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(2) (a) Show that $\cos(3x) \equiv 4\cos^3 x - 4\cos x$

(b) Hence, or otherwise, solve the equation $4\cos^3 3y - 4\cos 3y = 1, \quad 0 \le y \le \frac{\pi}{4}$

(3) Given that $12\sin x \equiv k\sin y\cos y$, find a possible value of the constant k and hence express y in terms of x.

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(38) $a\cos x \pm b\sin x$ as $R\cos(x \pm \alpha)$ $R\sin(x \pm \alpha)$

WORKING AT D/E

(1) (a) Use the formula book to find the expansion of $\sin(x+\alpha)$

(b) Hence, write down the expansion of $R\sin(x+\alpha)$

(c) Using your answers to part (a) and (b), show that $3\cos x + 4\sin x$ can be written as $5\sin(x + 36.9^o)$

(2) (a) Using the expansion for $\cos(A+B)$ in the formula book, show that

$$12\cos x + 5\sin x = 13\cos(x - 22.6^o)$$

(b) Hence, show that the solutions to the equation $12\cos x + 5\sin x = 6.5$ are $x = 82.6^0$ and $x = 322.6^0$ for $0 < x < 360^o$

(3) Write $8\cos x - 6\sin x$ in the form $R\cos(x+\alpha)$ where $R > 0$ and $0 < \alpha < 90^o$

WORKING AT B/C

(1) (a) Express $8\cos x + 6\sin x$ in the form $R\sin(x+\alpha)$ where $R > 0$ and $0 < \alpha < 90^o$

(b) Hence, solve the equations below:

(i) $8\cos x + 6\sin x = 10,\ \ 0 \le x \le 360^o$

(ii) $8\cos 2y + 6\sin 2y = 5,\ \ 0 \le y \le 360^o$

(2) (a) $2\cos x + 4\sin x$ in the form $R\cos(x-\alpha)$ where $R > 0$ and $0 < \alpha < 90^o$. Give your answer for R in exact form and α to 1 decimal place.

(b) $f(x) = 2\cos x + 4\sin x,\ x \in R$. Using your answer to part (a), find the maximum value of $f(x)$.

(c) Write down the coordinates of the first maximum point of $f(x),\ x > 0$.

(d) Explain why there are no solutions to the equation $2\cos x + 4\sin x = 4.5$ for any value of x

(3) (a) Express $3\cos x + 2\sin x$ in the form $R\cos(x-\alpha)$ where $R > 0$ and $0 < \alpha < 90^o$

(b) Hence sketch the graph of $y = 8\cos x + 6\sin x,\ \ 0 \le x \le 360^o$ including the coordinates of any stationary points and points where the curve meets or crosses the coordinate axes.

WORKING AT A*/A

(1) $g(x) = 10 - \cos 5x + \sin 5x$

(a) Show that $g(x)$ can be written in the form

$$g(x) = p\sin(5x - q) + r\ , \quad p > 0,\ 0 < q < \frac{\pi}{2}$$

where p, q and r are constants.

(b) Find the maximum value of $g(x)$ in exact form.

(c) Given that $0 \le x \le \pi$, find the values of x that that maximise $g(x)$, giving exact values.

(d) Sketch the graph of $y = g(x),\ 0 \le x \le \pi$. On the graph show the coordinates of any maximum or minimum points and the coordinates of any points where the graph meets or crosses the coordinate axes. Any non-exact values are to be given to 3SF.

(2) (a) Express $7\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where $R > 0$ and $0 < \alpha < 90^o$.

$f(x) = \frac{5}{7\cos x - \sin x}$

(b) Find the least value of $|f(x)|$ as a simplified surd

(c) Express $f(x)$ in the form $P\sec(x+\alpha)$

(d) Hence, sketch the graph of $y = f(x)$ $0 \le x \le 360^o$ including any asymptotes.

(3) (a) Solve the equation $2\cos x - \sin x = \frac{\sqrt{5}}{2}$, $0 < x < 2\pi$, giving your answers to 3SF in radians.

(b) Hence, solve $\sin 4y - 2\cos 4y = \frac{\sqrt{5}}{2}$, $-\pi < y < \pi$, giving your answers in exact form.

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(39) Proving Trigonometric Identities

WORKING AT D/E

(1) Using the formula book, prove that $\cos(x-90) - \cos(x+90) \equiv 2\sin x$

(2) Prove $\frac{\sin 2A}{2\cos^2 A} \equiv \tan A$

(3) Show that $\frac{\cos 4A+1}{2} \equiv \cos^2 2A$

WORKING AT B/C

(1) (a) Prove that $\frac{1}{2}\sec x \sin 2x \equiv \sin x$

(b) Hence, solve the equation $\frac{1}{2}\sec y \sin 2y = \cos y$, $0 < y < 360^o$

(2) (a) Prove that $\operatorname{cosec} 2A \cos 2A \equiv \cot 2A$

(b) Hence, solve $\operatorname{cosec} 2x \cos 2x = \sqrt{3}$, $0 \leq x \leq 2\pi$, giving your answers as multiples of π.

(3) Prove that $\left(\frac{2\sin x\ cosx}{2\cos^2 x - 1}\right)^2 \equiv \sec^2 2x - 1$

WORKING AT A*/A

(1) Given that $\cos(2x+y) = 0$, show that $\tan y = \cot 2x$

(2) Solve the equation $2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 1,\ 0 < x < 2\pi$, giving your answers in exact form.

(3) (a) Prove that $(\cos 2x + \sin 2x)^2 \equiv 1 + \sin 4x$

(b) Hence, or otherwise, solve the equation $(\cos 2x + \sin 2x)^2 = 2\operatorname{cosec} 4x$, $0 < x < 360^o$

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(40) Applications of Trigonometric Functions

WORKING AT D/E

(1) A dot is moving backwards and forwards across the full length of a computer screen. The position ($P\ cm$) of the dot after (t) seconds relative to the centre of the screen can be modelled by the equation

$$P = 10\cos 20t^{o}, \quad t \geq 0$$

(a) Explain why the width of the computer screen is $20cm$.

(b) Find out where the dot was when it first started moving.

(c) Calculate the time when the dot was first 4cm to the left of the centre. Give your answer to 3SF.

WORKING AT B/C

(1) Cyril is making a mini wave machine for a science project. He places the device in a tank with water. The tank is in the shape of a cuboid. He positions a small rubber duck in the water and studies the height of the duck over a period of time. The duck's height ($H\ cm$) relative to the central height of the tank after (t) seconds can be modelled by the equation

$$H = 8\sin\left(4t + \frac{\pi}{2}\right) + 2, \ \ t \geq 0$$

where all angles are measured in radians.

(a) Show that the initial height of the duck is $6cm$ above the centre of the tank.

(b) Find the maximum possible height above the centre of the tank the duck reaches.

(c) Find how long it takes the duck to first reach this height giving your answer to 3SF..

(d) Given that the depth of the tank is $30cm$, explain why the duck never hits the bottom of the tank.

(e) Find the first time that the duck is at the central height of the tank to 3SF.

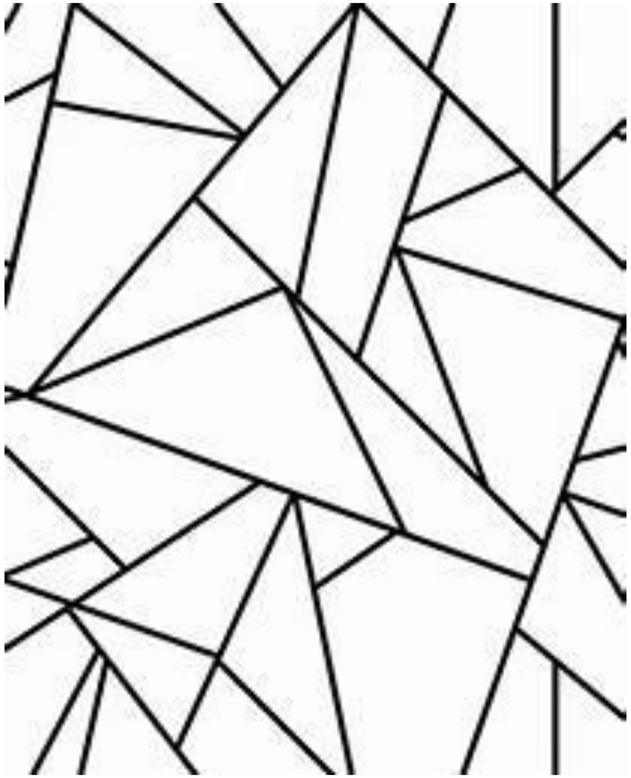
WORKING AT A*/A

(1) (a) Express $5\cos x + 2\sin x$ in the form $R\sin(x+\alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving your answer for R in exact form and α to 1 decimal. place.
A robot is programmed to follow a path that can be modelled by the equation

$$H = 5\cos\left(\frac{\pi t}{12}\right) + 2\sin\left(\frac{t\pi}{12}\right) + 4, \quad 0 \leq t \leq 10$$

where H is the height of the robot relative to a fixed point and t is the time in seconds after the robot starts the path.
(b) Find the maximum height of the robot.
(c) Find the first time the robot reaches this maximum height giving your answer to 3SF.
(d) Find to 2 decimal places, the time when the robot has a height of 5 metres above the fixed point.
(e) Find to 2 decimal places, the time when the robot has a height of 1 metre below the fixed point.

Parametric Equations



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(41) Parametric Equations Evaluating and Converting

WORKING AT D/E

(1) A curve has parametric equations:

$$x = t - 2, \qquad y = t^2, \qquad -8 \le t \le 4$$

(a) Find a cartesian equation for the curve in the form $y = f(x)$

(b) Show that the domain of $f(x)$ is $-10 \le x \le 2$

(c) Show that the range of $f(x)$ is $0 \le f(x) \le 64$

(d) Hence, sketch the graph of $y = f(x)$

(2) A curve has parametric equations:

$$x = \ln t, \qquad y = 4 - t, \qquad t > 0$$

(a) Show that the cartesian equation for the curve is $y = 4 - e^x$

(b) Explain why the domain of the cartesian equation is $x \epsilon R$.

(c) Show that the range of the cartesian equation is $y < 4$

(d) Hence, sketch the graph of $y = f(x)$

WORKING AT B/C

(1) A curve has parametric equations:

$$x = e^{2t}, \qquad y = 1 + t, \qquad t \in R$$

(a) Find a cartesian equation for the curve in the form $y = f(x)$

(b) Find the domain and range of the cartesian equation.

(2) A curve has parametric equations:

$$x = \frac{1}{t-3}, \qquad y = \frac{1}{t+9}, \qquad 4 < t < 5$$

(a) Show that the cartesian equation for the curve can be written as $y = \frac{x}{12x+1}$

(b) Find the domain of $f(x)$

(c) Hence, find the range of $f(x)$

(3) A curve has parametric equations:

$$x = \ln(t - 3), \qquad y = \frac{1}{t}, \qquad t > 4$$

(a) Find a cartesian equation for the curve in the form $y = f(x)$

(b) Find the domain and range of the cartesian equation.

WORKING AT A*/A

(1) A curve has parametric equations:

$$x = 2\sqrt{t}, \qquad y = 256t^3, \qquad t \ge 0$$

Find the cartesian equation in the form $y = Ax^n$ stating the domain and the range of the function.

(2) A curve has parametric equations:

$$x = e^{2t-1}, \qquad y = \sqrt{t}, \qquad t \ge 0$$

(a) Find the cartesian equation in the form $y = f(x)$

(b) State the domain of the cartesian equation in exact form.

(c) State the range of the cartesian function.

(3) A curve has parametric equations:

$$x = t - 4, \qquad y = -t^3, \qquad t \in R$$

(a) Find a cartesian equation for the curve in the form $y = f(x)$

(b) Hence, sketch the graph of $y = f(x)$ stating the domain and the range.

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(42) Trigonometric Identities for Parametric Equations

WORKING AT D/E

(1) A circle has parametric equations:

$$x = \cos\theta - 1, \qquad y = \sin\theta + 4,$$

(a) Using the identity $\cos^2\theta + \sin^2\theta \equiv 1$ find a cartesian equation for the circle.

(b) Hence, sketch the circle showing where the curve meets the coordinate axes.

(2) A curve has parametric equations:

$$x = \sec t, \qquad y = 2\tan t, \qquad 0 < t < \frac{\pi}{2}$$

(a) Use a trigonometric identity to show the cartesian equation of the curve is $y^2 = 4(x^2 - 1)$

(b) Explain why the range is $y > 0$

(3) A circle has parametric equations:

$$x = 5\cos\theta - 2, \qquad y = 5\sin\theta + 3,$$

(a) Find a cartesian equation for the circle.

(b) Write down the length of the radius of the circle.

WORKING AT B/C

(1) A curve has parametric equations:

$$x = \cos\theta + 1, \qquad y = \sin 2\theta, \qquad 0 < \theta < \pi$$

(a) Show that the cartesian equation can be written as $y = 2(x-1)\sqrt{x(2-x)}$

(b) Find the domain and the range of the function.

(2) A curve has parametric equations:

$$x = \tan^2 t, \qquad y = \frac{3}{\sin^2 t}, \qquad 0 < t < \frac{\pi}{2}$$

(a) Find a cartesian equation for the curve in the form $y = f(x)$.

(b) Find the domain and the range of $f(x)$

(3) A curve has parametric equations:

$$x = \cos 2t, \qquad y = \sin t, \qquad 0 < t \leq 2\pi$$

Find a cartesian equation in the form $x = f(y)$

WORKING AT A*/A

(1) A curve has parametric equations:

$$x = 4\cos t, \qquad y = \sin\left(t - \frac{\pi}{3}\right), \qquad 0 < \theta < \pi$$

(a) Show that the cartesian equation can be written as $y = \frac{1}{16}\left[2\sqrt{16 - x^2} - \sqrt{3}x\right]$

(b) Find the domain and the range of the function.

(2) A curve has parametric equations:

$$x = \cos 2t, \qquad y = \cot 2t, \qquad 0 < t < \frac{\pi}{4}$$

(a) Find a cartesian equation for the curve in the form $y^2 = f(x)$.

(b) Find the domain and the range of $f(x)$

(3) A cartesian equation is given by

$$(x+7)^2 + (y-3)^2 = 16$$

Write down the parametric equations of the circle in the form $x = f(t)$ and $y = f(t)$ stating a suitable domain for t.

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(43) Sketching Parametric Curves

WORKING AT D/E

(1) A curve has parametric equations:

$$x = 2t, \quad y = t^2, \quad -2 \le t \le 3$$

(a) Complete the table below

t	-2	-1	0	1	2	3
x						
y						

(b) Hence, **plot** the curve given by the parametric equations: $x = 2t,\ y = t^2, -2 \le t \le 3$ on graph or squared paper.

(2) A curve has parametric equations:

$$x = 3\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$$

(a) Complete the table below

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x					
y					

(b) Hence, **plot** the curve given by the parametric equations: $x = 3\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$ on graph or squared paper.

WORKING AT B/C

(1) A curve has parametric equations:

$$x = \cos 2t, \quad y = \sin t, \quad 0 \le t \le \frac{\pi}{2}$$

(a) Construct a table to find the values of x and y for common trigonometric values of t, for $0 \le t \le \frac{\pi}{2}$

(b) Hence, **plot** the curve given by the parametric equations: $x = \cos 2t,\ y = \sin t,\ 0 \le t \le \frac{\pi}{2}$ on graph or squared paper.

(2) A curve has parametric equations:

$$x = 1 - \cos t, \quad y = 1 + \sin t, \quad 0 \le t \le 2\pi$$

Plot the curve on squared paper.

(3) A curve has parametric equations:

$$x = \frac{1}{t}, \quad y = 0.5t^2, \quad 1 \le t \le 4$$

Plot the curve on squared paper.

WORKING AT A*/A

(1) A curve has parametric equations:

$$x = 2\sec t, \quad y = \tan t, \quad -\frac{\pi}{4} \le t \le \frac{\pi}{4}$$

Plot the curve on squared paper showing the coordinates in exact form where appropriate.

(2) A curve has parametric equations:

$$x = 4 + 9\cos 2t, \quad y = 9\sin 2t - 3, \quad 0 \le t \le \pi$$

(a) Find the cartesian equation of the curve.

(b) Hence, sketch the curve.

(c) Find the length of the curve in exact form.

(3) A curve has parametric equations:

$$x = 3\cos 6t, \quad y = 2\sin 6t, \quad 0 \le t \le \pi$$

(a) Show that the curve does not form part of a circle.

(b) Sketch the curve showing where it meets or crosses the coordinate axes.

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(44) Points of Intersection of Parametric Curves

WORKING AT D/E

(1) A curve has parametric equations:

$x = 2t - 4, \quad y = t^2, \quad -6 \leq t \leq 3$

(a) The curve crosses the y axis at A. Find the value of t at the point A.

(b) Hence, find the coordinates of A.

(c) The curve crosses the x axis at B. Find the value of t at the point B.

(d) Hence, find the coordinates of B.

(2) A curve has parametric equations:

$x = 3t, \quad y = t^2, \quad -4 \leq t \leq 6$

Given that the line with catering equation $y = x + 10$ meets the curve at two points

(a) Show that $t^2 - 3t - 10 = 0$

(b) Find the two roots of the equation $t^2 - 3t - 10 = 0$.

(c) Hence, find the coordinates where the line with equation $y = x + 10$, meets the curve with parametric equations $x = 3t, y = t^2$

(3) A circle has parametric equations:

$x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq 2\pi$

Find the coordinates of the 4 points where the circle crosses the coordinate axes.

WORKING AT B/C

(1) A curve has parametric equations:

$x = e^{3t} - 1, \quad y = \ln(t - 3), \quad t > 3$

(a) Show that the curve doesn't intersect the y axis.

(b) Find the coordinates of where the curve crosses the x axis giving your answer in exact form.

(2) A curve has parametric equations:

$x = e^{2t} + 1, \quad y = e^t, \quad t \in R$

The line with equation $y = x - 13$ meets the curve at the point A. Find the coordinates of A.

(3) A curve has parametric equations:

$x = 4pt, \quad y = pt^2 - 8, \quad -2 \leq t \leq 2$

where p is a constant.

(a) Given that the point $(8, -6)$ lies on the curve, find the value of p.

(b) The curve crosses the x axis at A and B and the y axis at C. Find the coordinates of A, B and C.

WORKING AT A*/A

(1) A curve has parametric equations:

$x = \frac{2+t}{t-1}, \quad y = 5 + t, \quad t \in R, \ t \neq a$

(a) Write down the value of a.

(b) Find the coordinates of the points where the curve crosses the coordinate axes.

(c) Explain why the line $y = 6$ does not meet the curve.

(d) Show that when line with equation $y = 10 + x$ meets the curve, $t = \frac{7 \pm \sqrt{37}}{2}$.

(2) A curve has parametric equations:

$x = \cot 4t \quad y = \sin t, \quad \frac{\pi}{12} < t < \frac{\pi}{2}$

Find all of the points where the curve crosses the coordinate axes giving answers to 3SF where appropriate.

(3) A curve has parametric equations:

$x = \sec t \quad y = \cot 2t, \quad 0 < t < \frac{\pi}{2}$

A line intersects the curve at the points where $t = \frac{\pi}{6}$ and $t = \frac{\pi}{3}$.

Find **an** equation of the line in cartesian form.

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(45) Applications of Parametric Equations

WORKING AT D/E

(1) Doris is holding a fireworks display. She lets off a rocket. The flight of the rocket can be modelled by the parametric equations:

$$x = 2t, \qquad y = 3t - 0.25t^2, \qquad t \geq 0$$

where x is the horizontal distance and y is the vertical distance in metres from the ground. t is the time in seconds after the rocket is launched.
(a) Show that the rocket was launched from the ground.
(b) Find how long it takes the rocket to hit the ground.
(c) Using your answer to part (b), find the horizontal distance from where the rocket was launched to where it hits the floor.
(d) State one limitation of the model.

(2) Cyril has a drone. The drone is flown off a ledge 10m above the ground and Cyril guides the drone into land on the floor. The position of the drone relative to the ground can be modelled by the parametric equations

$$x = 8t, \qquad y = 10 - \frac{t^2}{10}, \qquad t \geq 0$$

Find the horizontal distance the drone travels from its starting point when it lands on the ground.

WORKING AT B/C

(1) A speed boat is moving around a buoy. Its horizontal displacement x metres and vertical displacement y metres relative to the buoy after t seconds can be modelled by the parametric equations:

$$x = 20\cos t, \qquad y = 20\sin t, \qquad t \geq 0$$

Where t is measured in degrees.
(a) Describe the motion of the boat.
(b) How far from the buoy is the boat at all times?
(c) Show that the boat is directly north of the buoy for the first time after approximately 1.57 seconds.

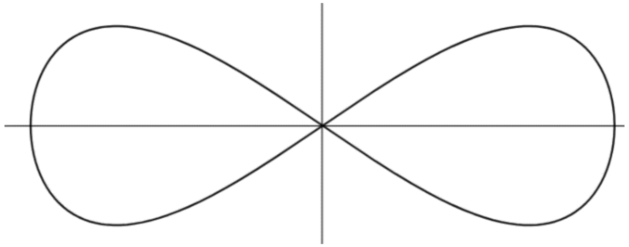
(2) A jet ski is moving around a buoy. Its horizontal displacement x metres and vertical displacement y metres relative to the buoy after t seconds can be modelled by the parametric equations:

$$x = 30\cos t, \qquad y = 20\sin t, \qquad t \geq 0$$

Where t is measured in degrees.
(a) Show that the boat is not on a circular path.
(b) How far from the buoy is the jet-ski at the start?
(c) Show that when the jet ski is due north of the buoy it is 20 metres away.
(d) Write down the maximum distance the jet-ski is from the buoy at any time.

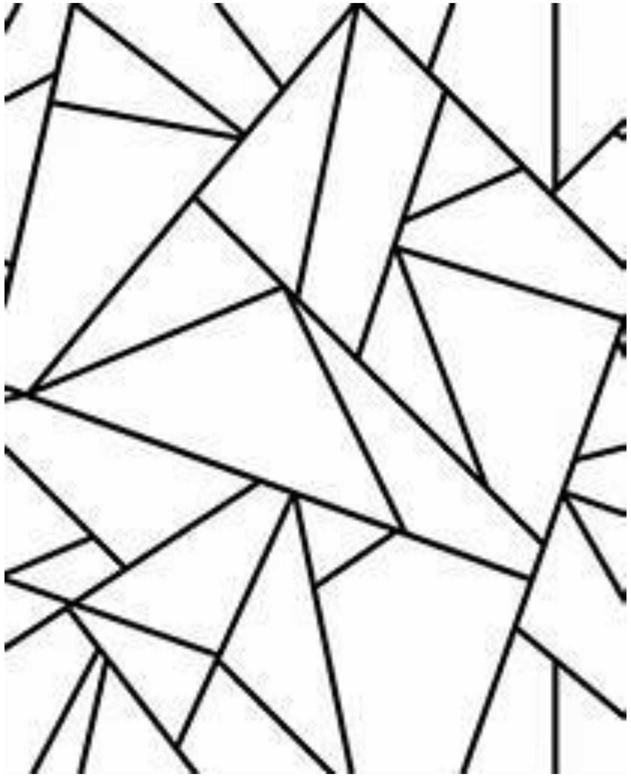
WORKING AT A*/A

(1) In a computer animation a buggy is racing around a track in the shape of a figure of 8 as shown below.



Its position (in cm) relative to the centre of the track after t seconds can be modelled by the parametric equations: $x = 20\cos 2t, \quad y = 10\sin 4t, \quad t \geq 0$
x and y are measured in radians.
(a) Mark on diagram the starting point of the buggy
(b) Find the first time the buggy is in the centre of the track.
(c) Find the time taken for the buggy to do one full figure of 8.

Differentiation



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(46) Differentiating $\sin x$ and $\cos x$ Functions

WORKING AT D/E

(1) Using the formula book, find $\frac{dy}{dx}$ for each of the following:

(a) $y = \cos 2x$ (b) $y = \sin 4x$ (c) $y = 5\cos 3x$
(d) $y = -6\sin 8x$ (e) $y = \sin(-3x) + 2\sin(4x)$

(2) $f(x) = 2\sin(x) - x, \quad 0 < x < \frac{\pi}{2}$
(a) Find an expression for $f'(x)$
The curve with equation $y = f(x)$ has a stationary point P.
(b) Show that the x coordinate of P is $\frac{\pi}{3}$
(c) Hence, show that the coordinates of P are $\left(\frac{\pi}{3},\ \sqrt{3} - \frac{\pi}{3}\right)$

(3) $g(x) = \cos 2x\,,\ \ 0 \le x \le \pi$
(a) Doris wants to find $g'(30^o)$. Can Doris do this?
(b) Find an expression for $g'(x)$
(c) Show that the gradient of the $g(x)$ at the point Q where $x = \frac{\pi}{4}$ is -2.
(d) Hence, show that the equation of the tangent at Q is $y = -2x + \frac{\pi}{2}$

WORKING AT B/C

(1) $f(x) = \sin x + \cos x\,,\ 0 \le x \le \pi$

(a) Show that the x coordinate of the stationary point on the curve of $y = f(x)$ satisfies the equation $\tan x = 1$.

(b) Hence, find the exact coordinates of the stationary point on the curve with equation $y = f(x)$.

(2) A curve has equation $y = \sin 2x - \cos 4x$.

(a) Find the equation of the tangent to the curve at the point $\left(\frac{\pi}{2}, -1\right)$

(b) Show that the tangent to the curve at the point $\left(\frac{\pi}{4},\ 2\right)$ is a horizontal line.

(3) Show that there are no stationary points on the curve with equation $y = \sin 2x - 3x,\ 0 \le x \le 2\pi$

WORKING AT A*/A

(1) Find the equation of the normal to the curve $y = 4\sin x \cos x$ at the point with x coordinate $\frac{\pi}{3}$ giving your answer in the form $ax + by = c$

(2) A curve has equation

$y = e^{2x} + 4\cos 6x\,,\ 0 < x \le \frac{\pi}{4}$.
Show that the x coordinate of the stationary point on the curve satisfies the equation
$x = \ln\sqrt{k\sin 2x}$
where k is an integer to be found.

(3) Prove, from first principles, that the derivative of $\sin x$ is $\cos x$

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(47) Differentiating Exponentials & Logs

WORKING AT D/E

(1) Find $\frac{dy}{dx}$ for each of the following:

(a) $y = e^x$ (b) $y = \ln x$ (c) $y = e^{3x}$

(d) $y = 2e^{-x}$ (e) $y = \ln 4x$ (f) $y = -4e^{0.5x}$

(2) Show that the equation of the tangent to the curve $y = e^{2x}$ at the point where the $x = 0$ is $y = 2x + 1$

(3) A curve has equation $y = x - 2\ln x$. Show that the coordinates of the stationary point on the curve are $(2,\ 2 - \ln 4)$

WORKING AT B/C

(1) Find $\frac{dy}{dx}$ for each of the following:

(a) $y = \frac{2}{e^x}$ (b) $y = \ln x^2$ (c) $y = e^{6x} - 2\ln x$

(d) $y = \frac{7 - e^{8x}}{e^{5x}}$ (e) $y = \ln \frac{1}{x}$ (f) $y = 4e^{\frac{x}{8}}$

(2) Show that the normal to the curve with equation $y = \ln 4x$ at the point with x coordinate 2 is $y = -2x + 3\ln 2 + 4$

(3) Find the coordinates of the stationary point on the curve with equation $y = e^{2x} - 8x$. Give your answer in exact form.

WORKING AT A*/A

(1) The tangent to the curve with equation $y = 2^x$ at the point $(0, p)$ crosses the x axis at Q.
(a) Write down the value of p.
(b) Find the coordinates of the point Q in exact form.

(2) The population of rats can be modelled by the equation $P = 300 \times 0.4^t$ where P is the number of rats after t days.
(a) Find the value of $\frac{dP}{dt}$ when $t = 8$.
(b) Interpret your answer in the context of the question
(c) State any limitations to the model.
(d) Cyril suggests a suitable domain for the function is $0 \leq t \leq 6$. Comment on his suggestion.

(3) (a) Find the coordinates of the stationary point on the curve with equation $y = \ln\left(\frac{1}{x^4}\right) + x^2,\ x > 0$ giving your answer in exact form.

(b) The normal to the curve at the point with x coordinate 1 crosses the x axis at A and the y axis at B.

Show that $AB = \frac{\sqrt{2}}{2}$

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(48) Differentiation using the Chain Rule

WORKING AT D/E

(1) Use the chain rule to find an expression for $\frac{dy}{dx}$ for each of the following.

(a) $y = (x^3 + 4)^6$ (b) $y = \cos^5 x$ (c) $y = 4\sin 2x$

(d) $y = \cos 8x$ (e) $y = e^{x^2}$ (f) $y = \ln(x^2 + 3x)$

(2) $y = e^{3x},\ x\epsilon R$

(a) Find an expression for $\frac{dy}{dx}$

(b) Find the value of y when $x = 1$ giving your answer in exact form.

(c) Hence, show that the equation of the tangent to the curve at the point where $x = 1$ is $y = 3e^3x - 2e^3$

(3) $y = \sin 3x + \cos 3x\ 0 \le x \le \frac{\pi}{2}$

(a) Find an expression for $\frac{dy}{dx}$

(b) Hence, show that any stationary points satisfy the equation $\tan 3x = 1$

(c) Show that the stationary points have x coordinates $x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$

WORKING AT B/C

(1) (a) $f(x) = \ln(2x^2 + 1),\ x > -1$

Show that the only stationary point on the curve is (0,0)

(b) $g(x) = \sqrt{3x^3 - x}\ ,\ -\frac{1}{\sqrt{3}} \le x \le 0$

(i) Show that $g'(x) = \frac{9x^2-1}{2\sqrt{3x^3-x}}$

(ii) Hence, find the only stationary point on the curve. Give your answer in exact form.

(2) A curve has equation $x = y^2 - y$

(a) Find an expression for $\frac{dx}{dy}$

(b) Hence, find the value of $\frac{dy}{dx}$ when $y = 3$

(3) A curve has equation $y = e^{x^2},\ x \in R$.

(a) Show that the only stationary point on the curve has coordinates (0,1)

(b) Show that the equation of the tangent to the curve at the point with x coordinate 1 is $y = 2ex - e$

WORKING AT A*/A

(1) (a) Given that $g(x) = \ln\cos^2 x\,, 0 \le x < \frac{\pi}{2}$ show that $g'(x) = k\tan x$, where k is a constant to be found.

(b) Hence, find the coordinates of the stationary point on the curve of $y = g(x)$.

(2) $y = e^{sin3x},\ 0 \le x \le \frac{\pi}{2}$

Find the exact coordinates of any stationary points on the curve.

(3) A curve has equation $y = \ln(x^2 + 6x)\,,\ x > -6$

(a) Show that the equation of the tangent to the curve at the point where $x = 1$ can be written as $y = \frac{8}{7}x + \ln 7 - \frac{8}{7}$

(b) Show that $f(x) = \ln(x^2 + 6x)\,,\ x > -6$ is an increasing function for all values of x.

(c) Find the only root of the equation $f(x) = 0$ giving your answer in exact form.

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(49) Differentiation using the Product Rule

WORKING AT D/E

(1) Circle all of the equations below where Cyril can use the product rule to find $\frac{dy}{dx}$.

(a) $y = x\cos x$ (b) $y = 3xe^{4x-2}$ (c) $y = x^2$

(d) $y = \sin(x)\sqrt{1-x^4}$ (e) $y = \ln(2x+7) + 3x$

(2) Using the formula book, or otherwise, find $\frac{dy}{dx}$ for each of the following using the product rule:

(a) $y = xe^x$ (b) $y = x^2\cos x$ (c) $y = \sqrt{x}\sin x$

(3) A curve has equation $y = 3x\cos 2x,\ x\epsilon R$

(a) Show that $\frac{dy}{dx} = 3\cos 2x - 6x\sin 2x$

(b) Hence, show that the gradient of the tangent to the curve at the point where $x = 0$ is 3.

WORKING AT B/C

(1) A curve has equation $y = x(x-1)^{\frac{1}{2}},\ x > 1$

(a) Show that $\frac{dy}{dx} = \frac{3x-2}{2(x-1)^{\frac{1}{2}}}$

(b) By considering the domain, explain why there are no stationary point on the curve.

(2) $y = 2 + e^x\sin x,\ \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$

(a) Find the equation of the tangent to the curve at the origin.

(b) Show that the stationary point on the curve satisfies the equation $\tan x = -1$

(c) Hence, find the exact coordinates of the stationary point.

(3) $f(x) = x\ln x,\ x > 0$

(a) Find an expression for $f'(x)$

(b) Hence, show that $f'(e^4) = 5$

WORKING AT A*/A

(1) A curve has equation $y = (x)\left(\sqrt[3]{x^2+1}\right),\ x\epsilon R$.

(a) Find the gradient of the curve at the point where $x = \sqrt{7}$ giving your answer in exact form.

(b) Explain why there is only one real root to the equation.

(2) $g(x) = 3e^{2x}\cos 2x,\ 0 < x < \frac{\pi}{2}$

Find the set of values of x such that $g(x)$ is an increasing function.

(3) A curve has equation $y = e^{-x}\ln x,\ x > 0$

(a) Show that $\frac{dy}{dx} = e^{-x}\left(\frac{1}{x} - \ln x\right)$

(b) By drawing two different graphs, write down the number of stationary points on the curve with equation $y = e^{-x}\ln x$

(c) Find the equation of the normal to the curve at the point where $x = 1$ giving your answer in the form $ax + by = c$.

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(50) Differentiation using the Quotient Rule

WORKING AT D/E

(1) For each of the following, state which rule would be used to differentiate the function. Write Product (P), Quotient (Q), Chain (C) or None of these (N).

(a) $f(x) = \frac{x}{\sin x}$ (b) $f(x) = xe^x$ (c) $f(x) = \frac{\ln x}{x}$

(d) $f(x) = (2x-4)^{-5}$ (e) $f(x) = e^{3x}$

(2) Using the formula book, or otherwise,

(a) show that if $y = \frac{x}{\ln x}$, $x > 1$ then $\frac{dy}{dx} = \frac{\ln x - 1}{[\ln x]^2}$

(b) Hence, show that there is a stationary point when $x = e$ and $y = e$

(3) Using the formula book, find an expression for $g'(x)$ for each giving your answer in its simplest form.

(a) $g(x) = \frac{x}{\cos x}$ (b) $g(x) = \frac{x^2}{e^{3x}}$ (c) $g(x) = \frac{\ln x}{\sin x}$

WORKING AT B/C

(1) A curve has the equation $y = \frac{2x-1}{e^x}$, $x \in R$.

(a) Find a fully simplified expression for $\frac{dy}{dx}$

(b) Hence, an equation for the normal to the curve at the point where $x = 1$

(2) $y = \frac{x^3+1}{\cos x}$, $0 \leq x < \frac{\pi}{2}$

Find $\frac{dy}{dx}$

(3) $f(\theta) = \frac{\theta}{\sin\theta}$, $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

(a) Find $f'(\theta)$

(b) Hence, show the stationary point on the curve satisfies the equation $\theta = \tan\theta$

(c) Find $f'\left(\frac{\pi}{2}\right)$

(d) Hence, show that the equation of the tangent to the curve when $\theta = \frac{\pi}{2}$ is $y = \theta$

WORKING AT A*/A

(1) $y = \frac{x}{\sqrt{x-1}}$, $x > 1$

(a) Show that $\frac{dy}{dx} = \frac{x-2}{2(x-1)^{\frac{3}{2}}}$

(b) Find the equation of the tangent to the curve at the point where $x = 3$ giving your answer in the form $ax + by = c$ where a and c are integers and b is in exact form.

(2) Find the exact coordinates of any stationary point on the curve with equation $y = \frac{x^2}{e^{3x}}$, $x \epsilon R$

(3) $g(x) = \frac{2}{x} + \frac{3}{x^2+x}$, $x > 0$

(a) Show that $g(x) = \frac{2x+5}{x^2+x}$

(b) Prove that there are no stationary points on the curve of $y = g(x)$

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(51) Differentiating other Trigonometric Functions

WORKING AT D/E

(1) Use the formula book to differentiate each of the following:

(a) $\tan x$ (b) $\sec x$ (c) $\cot x$
(d) $\operatorname{cosec} x$ (e) $\tan 2x$ (f) $-\sec 4x$

(2) Given that $y = \sec^2 x$ use the chain rule to show that $\frac{dy}{dx} = \sec^2 x \tan x$

(3) Use either the product or quotient rule to differentiate each of the following:

(a) $x \tan x$ (b) $\frac{\tan x}{e^x}$ (c) $e^{3x} \operatorname{cosec} x$

(d) $\frac{\cot 4x}{3x}$

WORKING AT B/C

(1) By writing $\tan x$ in terms of $\sin x$ and $\cos x$, use the quotient rule to show that $\frac{d}{dx}\tan x = \sec^2 x$

(2) (a) Show that $\frac{3}{\sin x \cos x} \equiv 6 \operatorname{cosec} 2x$

(b) Hence, find the derivative of $\frac{3}{\sin x \cos x}$

(3) $y = e^{\sin x},\ x \epsilon R$

(a) Find $\frac{dy}{dx}$

(b) Hence, show that the tangent to the curve where $x = 0$ is $y = x + 1$

WORKING AT A*/A

(1) (a) By writing $\cot x$ in terms of $\sin x$ and $\cos x$, find $\frac{d}{dx}\cot x$.

(b) $y = \cot x,\ 0 < x < \pi$

Find the equation of the tangent to the curve at the point where $x = \frac{\pi}{6}$ in the form $y = mx + c$

(2) (a) $x = \sin 2y,\ \frac{-\pi}{4} < y < \frac{\pi}{4}$

(b) Find $\frac{dy}{dx}$ in terms of y

(c) Find $\frac{dy}{dx}$ in terms of x

(3) (a) Show that $\frac{(\sin x + \cos x)^2}{\sin 2x} = 1 + \operatorname{cosec} 2x$

(b) Hence, find $\frac{d}{dx}\ \frac{(\sin x + \cos x)^2}{\sin 2x}$

(c) Given that $g(x) = \frac{(\sin x + \cos x)^2}{\sin 2x}$, $0 < x < \frac{\pi}{6}$, show that $g(x)$ is never stationary.

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(52) Differentiating Parametric Equations

WORKING AT D/E

(1) Find an expression for $\frac{dy}{dx}$ in terms of the parameter t for each of the following pairs of parametric equations:

(a) $x = 3t,\ y = t^2$ (b) $x = e^{2t}, y = \cos t$

(c) $x = \sqrt{t}\,, y = 4t^3$ (d) $x = \cos t, y = \sin t$

(2) A curve has parametric equations
$x = -\cos t\,,\quad y = \sin t\,,\quad 0 < t < \frac{\pi}{2}$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(b) Hence, show that the equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$ is $y = x - \sqrt{2}$

(3) A curve has parametric equations
$x = \ln t\,,\ y = t^2,\ t > 0$

(a) Show that $\frac{dy}{dx} = 2t^2$

(b) Hence, find the equation of the tangent at the point where $t = 1$

WORKING AT B/C

(1) A curve has parametric equations
$x = 8 - t^2,\ y = t^3,\ t\epsilon R$

Find the equation of the normal to the curve at the point where $t = 4$ in the form $ax + by + c = 0$

(2) The curve C has parametric equations
$x = 2\sin t - t\,,\quad y = \cos t + 3,\quad 0 < t < \frac{\pi}{2}$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(b) Hence, find the coordinates of the stationary point on the curve.

(3) A curve has parametric equations
$x = \ln t\,,\ y = t^2 - 8t,\ t > 0$

Show that the only stationary point on the curve has coordinates $(2\ln 2,\ -16)$

WORKING AT A*/A

(1) A curve has parametric equations
$x = 3\cos 4t - 4\,,\quad y = \sin 2t + 3,\quad 0 < t < \frac{\pi}{6}$

(a) Show that $\frac{dy}{dx} = k\operatorname{cosec} 2t$ where k is a constant to be found.

(b) Show that there is a point on the curve where the tangent has a gradient of $-\frac{1}{6}$.

(2) The curve C has parametric equations

$x = 2\cos\frac{t}{2}\,,\ y = 1 - \sin 2t\,,\ \ 0 \le t \le \pi$

Point P on the curve C has coordinates $(\sqrt{2}, 1)$

(a) Find the value of t at the point P.

(b) Find the equation of the tangent to the curve at the point P in the form $ax + by = c$

(3) A curve has parametric equations
$x = 4t^2,\ y = t^2 - 8t,\ t\epsilon R$

(a) Find an equation for the tangent to the curve at the point where $t = 0.5$

(b) Prove that this is the only point the where the tangent meets the curve.

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(53) Implicit Differentiation

WORKING AT D/E

(1) (a) Given that $y + 2x - 3y^2 = x^3 + 8$, show that $\frac{dy}{dx} = \frac{3x^2 - 2}{1 - 6y}$

(b) Given that $\sin y + \cos x = x - 3y$, show that $\frac{dy}{dx} = \frac{1 + \sin x}{3 + \cos y}$

(2) Find $\frac{dy}{dx}$ given that $\frac{y^3}{3} + \frac{1}{x} = x + 2y$ giving your answer in terms of x and y.

(3) Find $\frac{dy}{dx}$ given that $\ln y - e^{2x} - x = 4y$ giving your answer in terms of x and y.

WORKING AT B/C

(1) (a) Given that $x^2 - 2xy = y^3$, show that $\frac{dy}{dx} = \frac{2(x-y)}{(3y^2+2x)}$, $x < 0$, $y < 0$

(b) Hence, show that any stationary points on the curve satisfy the equation $y = x$

(c) Using your answer to part (b), show that there is a stationary point when $x = -1$

(2) Find $\frac{dy}{dx}$ when $\cos x - 3\sin 2y = 0.5$ giving your answer in terms of x and y

(3) (a) Find $\frac{dy}{dx}$ when $\frac{x^2}{y} + x = 6$

(b) Hence, show that the equation of the tangent to the curve with equation $\frac{x^2}{y} + x = 6$, where $x > 0$, $y > 0$ at the point (2,1) is $y = \frac{5}{4}x - \frac{3}{2}$

WORKING AT A*/A

(1) $2\cos x - \tan y = 1$, $0 \leq x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(b) Hence, find the equation of the tangent at the point $\left(\frac{\pi}{3}, \frac{\pi}{4}\right)$

(c) Find the coordinates of the stationary point on the curve.

(2) A curve has equation $xe^{4y} - 16x = y + 1$, $x > 0, y > 0$

(a) Show, that if the curve is stationary, the y coordinate of the stationary point is $\ln 2$

(b) Hence, show there are no stationary points on the curve.

(3) Find a simplified expression for $\frac{dy}{dx}$ given that $\sin(x + y) = 0$

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(54) Using the 2nd Derivative

WORKING AT D/E

(1) $f(x) = -x^2 + 2,\ x \in R$

(a) Find $f'(x)$

(b) Find $f''(x)$

(c) Hence, show that $f(x)$ is concave for all values of x.

(2) $g(x) = 8x + 4x^2\ x\epsilon R$

By considering $g''(x)$, show that the curve is convex for all values of x.

(3) $f(x) = e^x,\ x\epsilon R$

(a) Sketch the graph of $y = f(x)$

(b) Prove that the curve is convex for all values of x

WORKING AT B/C

(1) $f(x) = \ln x + 3x^2,\ x \geq 1$

(a) Show that $f''(x) = 6 - \frac{1}{x^2}$

(b) By considering the domain of $f(x)$, explain why the function is convex for all values x.

(2) $g(x) = 2x - e^{3x},\ x\epsilon R$

(a) Find $g'(x)$

(b) Find $g''(x)$

(c) State whether $g(x)$ is concave or convex for all values of x giving a justification for your answer.

(3) $f(x) = x^3 - 2x^2 - 11x + 12,\ x\epsilon R$

(a) Show that $f(1) = 0$

(b) Use polynomial division to express $f(x)$ in the form $f(x) = (x + a)(x + b)(x + c)$

(c) Hence, sketch the graph of $y = f(x)$

(d) Show that the graph is convex when $x \geq \frac{2}{3}$ and concave when $x \leq \frac{2}{3}$

WORKING AT A*/A

(1) $f(x) = (\cos x + \sin x)^2,\ 0 \leq x \leq \pi$

Use calculus to show that the function is concave when $\frac{\pi}{2} \leq x \leq \pi$

(2) $g(x) = 3x + \frac{1}{x}\ , x \neq 0$

Find the values of x for which the curve of $y = g(x)$ is concave.

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(55) Rates of Change (Differentiation)

WORKING AT D/E

(1) Given that $y = x^3 + 2x^2$ and that $\frac{dx}{dt} = 4$, show that $\frac{dy}{dt} = 28$ when $x = 1$.

(2) Given that $y = e^x + \sin x$ and that $\frac{dx}{dt} = -1$, show that $\frac{dy}{dt} = -2$ when $x = 0$.

(3) Given that $A = \pi r^2 + 4\pi r$ and $\frac{dA}{dt} = 6$, find the value of $\frac{dr}{dt}$ when $r = 1$ giving your answer to 3 SF.

WORKING AT B/C

(1) A sphere with radius r has volume , $V = \frac{4}{3}\pi r^3$
The volume of the sphere is increasing at a constant rate of $16cm^3s^{-1}$

Find the rate of change of the radius when the radius is $2cm$.

(2) A curve has equation $y = f(x),\ x > 0$. The gradient of the curve at any point on the curve is proportional to xy.

Point P on the curve has coordinates $(3, -4)$ and the gradient at P is $\frac{1}{3}$..

Show that $\frac{dy}{dx} = -\frac{xy}{36}$

(3) A circle has area A, circumference C and radius r. The area of the circle is increasing at a constant rate of $6cm^2s^{-1}$

(a) Find a formula for A in terms of r.

(b) Find a formula for C in terms of r.

(c) Show that the rate of change of the circumference when the radius is $4cm$ is $1.5cms^{-1}$

WORKING AT A*/A

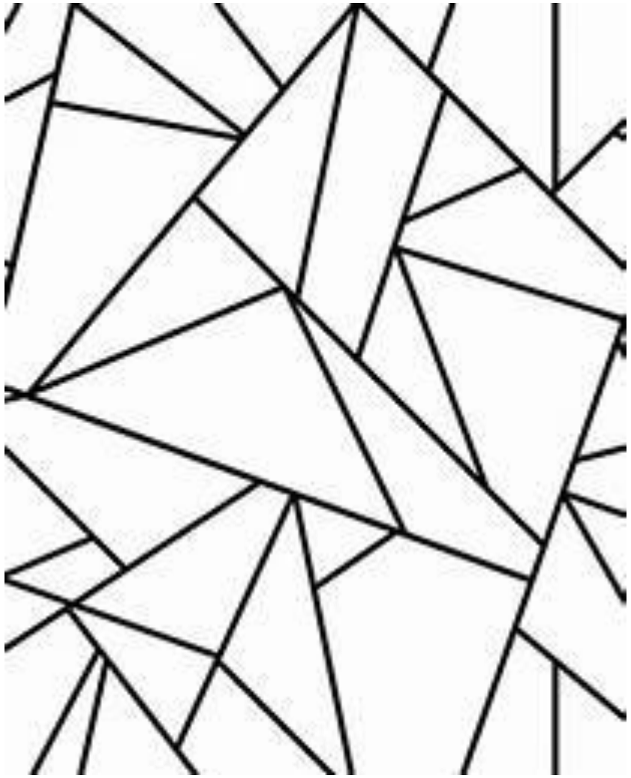
(1) A solid ice shaped cylinder with height $10cm$ and base radius r is melting at a constant rate of $1cm^3s^{-1}$

Show that an expression for the rate at which the surface area of the ice shaped cylinder is decreasing can be written as $\frac{5+r}{5r}$

(2) A solid cube has volume V and surface area A. After t seconds the volume of a solid cube is increasing at the rate of $8cm^3s^{-1}$.

Show the rate at which the area increases satisfies the differential equation $A^{0.5}\left(\frac{dA}{dt}\right) = 32\sqrt{6}$

Numerical Methods



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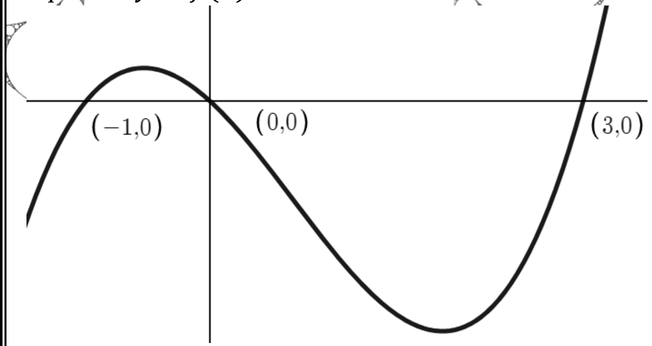
(56) Numerical Methods Locating Roots

WORKING AT D/E

(1) $f(x) = x(x+1)(x-3),\ x \in R$

The diagram below shows part of the curve with equation $y = f(x)$



(a) Write down the roots of the equation.

(b) Find the value of (i) $f(2.9)$ (ii) $f(3.1)$

(c) Explain how your answers in part (b) confirm that there is root in the interval (2.9, 3.1)

(2) $g(x) = \ln(x-5),\ x > 5$

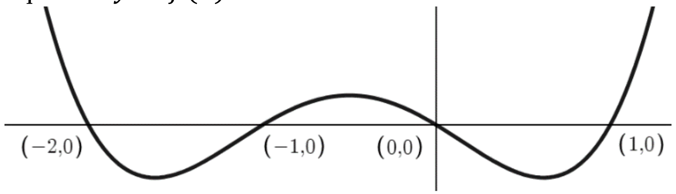
(a) Find the value of (i) $f(5.9)$ (ii) $f(6.1)$

(b) What does your answer to part (b) tell you?

(3) Doris wants to locate a root of the equation $h(x) = \frac{1}{x}$. She finds the value of $h(-0.1)$ and $h(0.1)$. Comment on her method.

WORKING AT B/C

(1) The diagram below shows part of the curve with equation $y = f(x)$

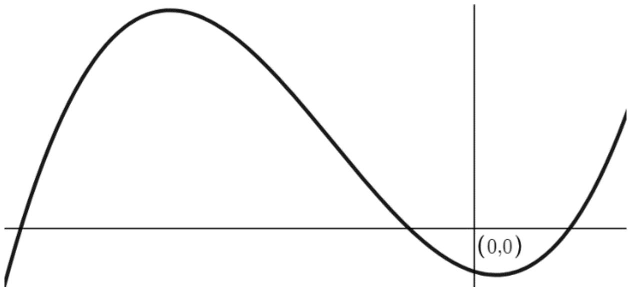


Cyril finds the value of $f(-1.1)$ and $f(0.1)$. He deduces that there is not root in the interval $-1.1 < x < 0.1$

(a) Why do you think he made this deduction?

(b) Comment on his findings.

(2) The diagram below shows part of the curve with equation $y = x^3 + 3x^2 - x - 1$



Show that there is a root of the equation in the interval [0.5,0.75]

(3) (a) Sketch the graphs of $y = x$ and $y = \cos x, -2\pi \leq x \leq 2\pi$ on the same set of axes.

(b) Using your sketch, write down the number of roots to the equation $x = \cos x$

(c) Show that there is a root is in the interval (0.7,0.8)

WORKING AT A*/A

(1) $f(x) = e^{2x} - \ln x, x > 0$

(a) Show there is a stationary point in the interval [0.28,0.29]

(b) Hence, determine the nature of the stationary point.

(c) The x coordinate of the stationary point is α. Show that $\alpha = 0.284$ correct to 3 decimal places.

(2) (a) On the same set of axes, sketch the graphs of $y = \sin x$ and $y = e^{-x}$, $0 < x < \pi$

(b) Using your answer to part (a) state the number of roots to the equation $e^x = \operatorname{cosec} x$

(c) Show that a root to the equation $e^x = \operatorname{cosec} x$ lies in the interval $0.5 < x < 0.6$

(3) $f(x) = e^x + \tan x,\ \frac{-\pi}{2} < x < \frac{\pi}{2}$.

(a) $f(x)$ has a root α. Show that $-0.6 < \alpha < -0.5$

(b) Cyril wants to find more roots to the equation outside the original domain. He finds that $f(1.5)$ is positive and $f(1.6)$ is negative. He says there must be a root in this interval. Explain he wrong.

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(57) Numerical Methods Iteration to Locate Roots

WORKING AT D/E

(1) $f(x) = x^2 - 4x + 1,\ x\epsilon R$
(a) Show that there is a root α to $f(x)$ in the interval $[3.7, 3.8]$
(b) Using the iterative formula $x_{n+1} = \sqrt{4x_n - 1}$, $x_0 = 4$, to find the value of x_1, x_2 and x_3 giving your answers to 5 decimal places.
(c) State the suitability of the iterative formula for locating α.
(d) Prove that $\alpha = 3.732$ correct to 3 decimal places.
(e) Doris decides to use the use the iterative formula $x_{n+1} = \frac{{x_n}^2+1}{4}$ to locate the other root of the $f(x)$. Taking $x_0 = 1$, show that the other root $\beta \approx 0.2679$

(2) $f(x) = x^5 + 5x^2 - 3,\ x\epsilon R$

(a) Show that there is a root α to the equation $x^5 + 5x^2 - 3 = 0$ in the interval $0.7 < x < 0.8$

(b) Show that $f(x) = 0$ can be written as $x = \sqrt{\frac{3-x^5}{5}}$
(c) Taking $x_0 = 0.5$, use the iterative formula $x_{n+1} = \sqrt{\frac{3-{x_n}^5}{5}}$ to find the values of x_1, x_2 and x_3 giving your answers to 5 decimal places.

(d) Prove that $\alpha = 0.744$ correct to 3dp.

WORKING AT B/C

(1) $f(x) = x - \sin 2x,\ x \in R$
(a) Show that there is a root α to $f(x)$ in the interval $[0.9, 1.0]$
$f(x) = 0$ can be written as either:
(i) $x = \frac{\sin^{-1} x}{2}$ (ii) $x = \sin 2x$
(b) Explain why using the iterative formula $x_{n+1} = \frac{\sin^{-1} x_n}{2}$, with $x_0 = 0.8$ doesn't locate α.
(c) Using the iterative formula $x_{n+1} = \sin 2x_n$ with $x_0 = 0.8$, find the values of x_1, x_2 and x_3 giving your answers to 5 decimal places.
(d) Prove that $\alpha = 0.9477$ correct to 4 S.F.

(2) $f(x) = 2x^2 - 2x + e^{5x},\ xeR$
(a) Find $f'(x)$
$f(x)$ has a stationary point β.
(b) Use your answer to part (a) to show that the x coordinate of β is in the interval $(-0.2, -0.1)$
(c) Use the iterative formula $x_{n+1} = 0.2 \ln\left(\frac{2-4x_n}{5}\right)$ with $x_0 = -0.1$ to find the values of x_1, x_2, x_3, x_4, x_5 and x_6 giving your answers to 5 decimal places.
(d) Prove that the stationary point on the curve with equation $y = f(x)$ has x coordinate -0.135 correct to 3 S.F
(e) Explain why the iterative formula $x_{n+1} = \frac{2-5e^{5x_n}}{4}$, $x_0 = -0.1$ doesn't locate the x coordinate of β

WORKING AT A*/A

(1) (a) On the same set of axes, sketch the graphs of $y = 1 - \ln(x+1),\ x > -1$ and $y = x,\ x \in R$
(b) Explain why there is one root to the equation $x = 1 - \ln(x+1)$
The function $f(x) = \ln(x+1) + x - 1,\ x > -1$
(c) Show that there is a root α to $f(x)$ in the interval $[0.5, 0.6]$
(d) Using the iterative formula $x_{n+1} = 1 - \ln(x_n + 1),\ x_0 = 0.5$ find the values of x_1, x_2, x_3, x_4, x_5 and x_6 giving your answers to 5 decimal places.
(e) Using your answer to part (d) explain why the iterations found in part (c) create a cobweb diagram.

(2) (a) On the same set of axes, sketch the graphs of $y = e^{4x} - 3$ and $y = x$
$f(x) = e^{4x} - 3 - x,\ x \in R$
(b) Using your answer to part (a), explain why there is a root to the equation $f(x)$ for $x > 0$
(c) Show that the root $0.2 < \alpha < 0.3$
(d) Show that the equation $f(x) = 0$ can be written as $e^{4x} - 3 = x$
(e) Doris uses the iterative formula $x_{n+1} = e^{4x_n} - 3$, $x_0 = 0.3$ to try and locate α. Find the values of x_1, x_2, x_3 and x_4 giving your answers to 5 decimal places.
(f) With the aid of a diagram, comment on the likely success of Doris' attempt.

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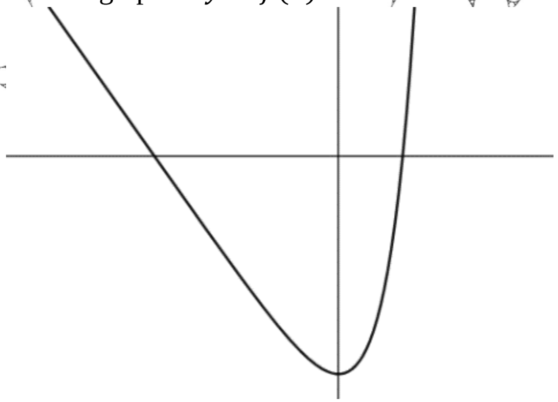
(58) Numerical Methods Newton-Raphson Method

WORKING AT D/E

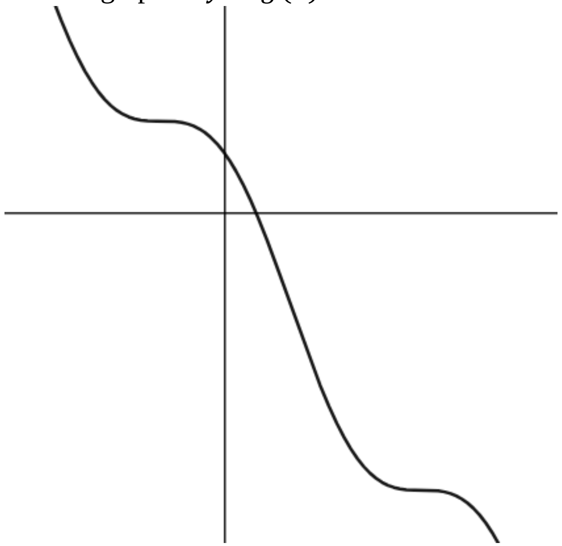
(1) $f(x) = e^x - x - 6, \quad x \epsilon R$
A root to the equation is α
(a) Show that $-6.0 < \alpha < -5.9$
Part of the graph of $y = f(x)$ is shown below



(b) Mark α on the diagram
(c) Find an expression for $f'(x)$
(d) **The Newton-Raphson formula is given in the formula book.** Using $x_0 = -5.5$ as an initial approximation for α, use the Newton-Raphson method to find x_1, $x_{2,}$, x_3 and x_4, giving each approximation to 6 dp.
(e) Show that $\alpha = -5.9975$ correct to 4 decimal places.
(f) β is the only other root of $f(x)$. Explain why $\beta > 0$.
(g) Mark on the diagram where $f'(x) = 0$

WORKING AT B/C

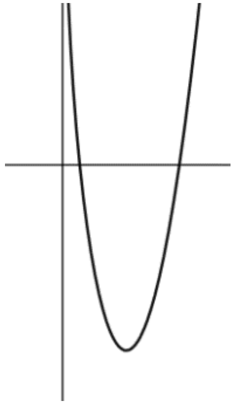
(1) $g(x) = \cos(x) - x, \quad x \epsilon R$ where x is measured in radians.
Part of the graph of $y = g(x)$ is shown below



(a) Mark any point on the diagram where it would not be appropriate to use the Newton-Raphson method to locate a root to the equation.
(b) Show that a root, α, of the equation $g(x) = 0$ is such that $0.7 < \alpha < 0.8$
(c) Using $x_0 = 0.85$ as an initial approximation for β, use the Newton-Raphson method to find x_1, $x_{2,}$, x_3 and x_4, giving each approximation to 4 dp.
(d) Show that $\alpha = 0.739$ to 3 SF.
(e) By considering the range of $\cos(x)$ explain why there are no more roots of $g(x)$
(f) Explain why there are more stationary points on the curve $y = g(x)$

WORKING AT A*/A

(1) $f(x) = 3x^2 - 4x - \ln 2x, \quad x > 0$
Part of the curve of $y = f(x)$ is shown in the diagram below.



The two roots to the equation are α and β where $\beta > \alpha$.
(a) Show that $0.2 < \alpha < 0.3$
(b) Show that $1.5 < \beta < 1.6$
(c) The curve has a stationary point in the interval $\gamma < x < \delta$. Write down possible values of γ and δ.
(d) Show that $f(x)$ is stationary in the interval $0.8 < x < 0.9$
(e) Cyril wants to find the root β to 3 decimal places. He decides to use the Newton-Raphson method to locate the root. He takes $x_0 = 0.9$. Comment on his approach.
(f) Using $x_0 = 1.3$ as an initial approximation for β, use the Newton-Raphson method to find x_1, $x_{2,}$, x_3 and x_4, giving each approximation to 4 dp.
(g) State a suitable starting value for x_0 to find an approximation for finding α using the Newton-Raphson method,
(h) Explain what the formula $x_{n+1} = x_n - \frac{6x-4-x_n^{-1}}{6+x_n^{-2}}$ could be used for.

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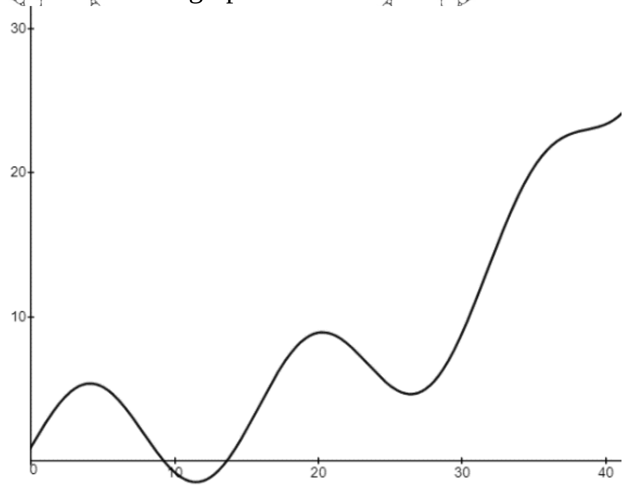
(59) Applications of Numerical Methods

WORKING AT D/E

(1) The flight of a bird is modelled by the equation $y = 4\sin(0.4t) + e^{0.08t}$ where y is the height in metres above sea leave of the bird at time t seconds after it takes off.
Approximately 40 seconds of the bird's flight is modelled in the graph below.

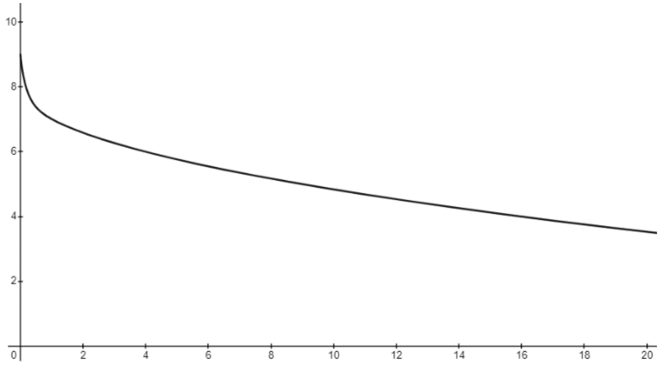


(a) Using the graph, estimate when the bird was first below sea level
(b) Prove this time was in the interval $9.2 < t < 9.3$
(c) Show that the bird returns to a height above sea level in the interval $13.6 < t < 13.7$
(d) Hence, **estimate** the length of time the bird was above sea leave in the first 40 seconds of its flight.

WORKING AT B/C

(1) The amount of a drug in a patient's body after it is first administered can be modelled by the equation $N = 8 - \sqrt{t} + e^{-0.01t}$ where N is the amount of the drug (in mg) and t is the number of hours after the drug is first given.
(a) Show that the initial dose of the drug was 9mg.

The diagram below shows the graph of the model for the first hours.

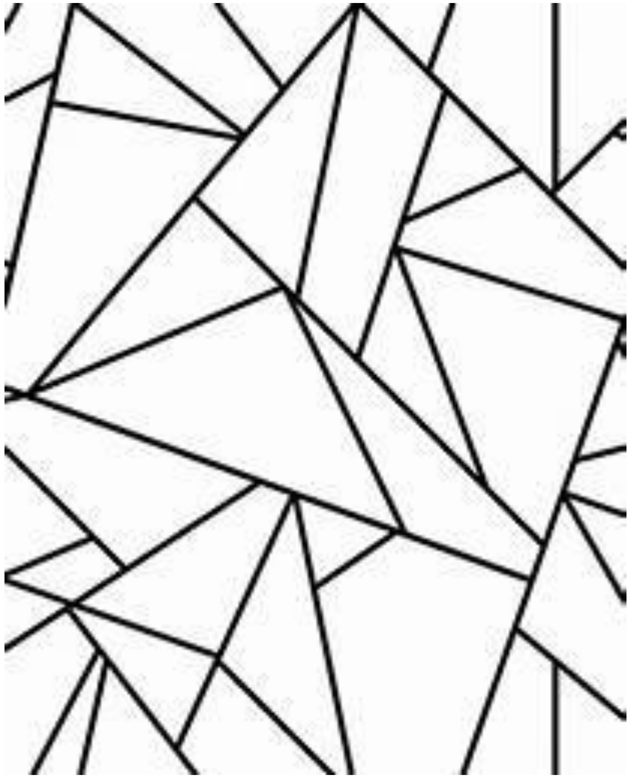


Doris wants to find out an approximation of how long it took before only 5mg of the drug remained in the patient's body.
(b) Show that Doris can use the equation $0 = -\sqrt{t} + e^{-0.01t} + 3$ to find an approximation for this time.
(c) Show that the time taken for there to be 5mg was between 14 hours and 54 minutes and 15 hours.
(d) Taking $t_0 = 15$ use the iterative formula $t_{n+1} = (e^{-0.01t_n} + 3)^2$ find t_1, t_2 and t_3 to 4 dp.
(e) Hence explain why the time taken was closer to 14 hours and 54 minutes than 15 hours.

WORKING AT A*/A

(1) A girl jumps from a diving board into a swimming pool. The height (H) of a girl above a swimming pool (t) seconds after she jumps can be modelled by the equation $H = 10 - t - 2^t$ where H is measured in metres.
(a) State the height above the water from which she jumps.
(b) Show that the girl hits the water between 2.5 and 3 seconds.
(c) Using 3.1 seconds as an initial approximation for the time the girl hits the water, use the Newton-Raphson method to find 4 further approximations for the time it takes to hit the water giving each approximation to 4 dp.
(d) Sketch the graph of H against t
(e) State one limitation of the model.

Integration



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(60) Integrating Standard Functions (Logs and Trig)

WORKING AT D/E

(1) **Use the formula book** to find the following integrals:

(a) $\int \sec^2 x \ dx$ (b) $\int \sec x \tan x \ dx$

(c) $\int \operatorname{cosec}^2 x \ dx$ (d) $\int -\operatorname{cosec} x \cot x \ dx$

(2) Considering the derivatives of $\sin x$, $\cos x$, $\ln x$ and e^x find the following integrals:

(a) $\int \cos x \, dx$ (b) $\int \sin x \ dx$

(c) $\int \frac{2}{x} dx$ (d) $\int e^{3x} \, dx$

(3) **Using the formula book** and the results above, find each integral below:

(a) $\int \tan x - \frac{1}{x} + e^x \, dx$

(b) $\int -2 \operatorname{cosec}^2 x - \sin x \ dx$

(c) $\int 4\sec^2 x - e^x - x \ dx$

(d) $\int \cot 2x - x^{-2} - \cos x \, dx$

WORKING AT B/C

(1) (a) Show that $\frac{4-x^2}{x} \equiv \frac{A}{x} + Bx$

(b) Hence, find $\int \frac{4-x^2}{x} \ dx$

(c) Simplify $\sin x\,(1 + \cot x)$

(d) Hence, find $\int \sin x\,(1 + \cot x)$dx

(2) (a) Find $\int \frac{5}{x} - \frac{1}{\sin^2 x} \ dx$

(b) (i) Show that $\frac{\sin x}{\cos^2 x} \equiv \sec x \tan x$

(ii) Hence, find $\int \frac{\sin x}{\cos^2 x} \ dx$

(3) Find the value of each giving your answers in exact form where appropriate. You must show full workings :

(a) $\int_0^{\frac{\pi}{4}} x - \sec^2 x \ dx$

(b) $\int_1^3 \frac{4}{x} - e^x + x \ dx$

(c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x - \operatorname{cosec}^2 x \ dx$

(d) $\int_1^6 \frac{8-3x}{x} \ dx$

WORKING AT A*/A

(1) Evaluate $\int_0^{\frac{\pi}{4}} \sec\theta\,(\sec\theta - \sin\theta)\, d\theta$ giving your answer in the form $\ln Ae$ where A is a simplified surd.

(2) Show that $\int_1^2 \frac{(1-x)^2}{2x} dx = A \ln B + \frac{C}{D}$ where A, B, C and D are rational constants to be found.

(3) Show that

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec} x\,(\cos x + \sin x)^2 \, dx = 2 - \sqrt{3} + \ln\sqrt{3}$$

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(61) Integrating Functions of the form $f(ax+b)$

WORKING AT D/E

(1) Find each of the following by **using the formula book**:

(a) $\int 2\sec 2x \tan 2x \, dx$ (b) $\int \tan 3x \, dx$

(c) $\int -4\operatorname{cosec}^2 2x \, dx$ (c) $\int 8 \operatorname{cosec} 4x \cot 4x \, dx$

(2) Show each of the following results. **You must show your full method:**

(a) $\int_0^1 e^{2x} dx = \frac{1}{2}(e^2 - 1)$

(b) $\int_0^{\frac{\pi}{2}} \cos\left(2x - \frac{\pi}{6}\right) dx = \frac{1}{2}$

(c) $\int_0^2 \frac{2x}{x^2+3} dx = \ln\frac{5}{3}$

(3) Show each of the following results:

(a) $\int_0^1 (3x+1)^2 dx = 7$

(b) $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} -2\operatorname{cosec} 2x \cot 2x \, dx = \frac{2\sqrt{3}}{3} - 2$

WORKING AT B/C

(1) Find $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$ showing full workings.

(2) Find each of the following integrals:

(a) $\int \frac{4x+2}{2x^2+2x} dx$

(b) $\int e^{3x} + \operatorname{cosec}^2 4x \, dx$

(c) $\int e^{2x+3} - \frac{3}{x} \, dx$

(d) $\int 4x + \sin(2-x) \, dx$

(3) Evaluate each of the following. You must show full workings and give answers in exact form:

(a) $\int_1^2 \frac{2x+3}{x^2+3x} \, dx$

(b) $\int_0^{\frac{\pi}{8}} -2\sin 4x \, dx$

(c) $\int_0^2 \frac{1}{3} e^{4x+1} \, dx$

.

WORKING AT A*/A

(1) (a) Show that

$(\operatorname{cosec} x + \tan x)^2 \equiv \operatorname{cosec}^2 x + 2\sec x + \sec^2 x - 1$

(b) Hence find $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\operatorname{cosec} x + \tan x)^2 dx$ giving your answer in exact form.

(2) Evaluate $\int_1^3 \frac{2x+7}{2x^2+14x} dx$ giving your answer in the form $\ln\frac{\sqrt{a}}{b}$ where $\sqrt{a}$ and b are in their simplest form.

(3) Given that $\int_1^{e^4} \frac{p}{x} dx = \frac{16}{3}$, where p is a positive constant, find the value of p.

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(62) Integrating using Trigonometric Identities

WORKING AT D/E

(1) (a) Write $\tan^2 x$ in terms of $\sec x$

(b) Hence, using the formula book, show that $\int_0^{\frac{\pi}{4}} \tan^2 x \, dx = 1 - \frac{\pi}{4}$

(2) (a) Using the formula book, show that $\cos(2x)$ can be written as $\cos^2 x - \sin^2 x$

(b) Hence, show that $\cos(2x) = 1 - 2\sin^2 x$

(c) Using your answer to part (b), show that $\int \sin^2 x \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

(3) (a) Using the formula book, write $\sin 2x$ in terms of $\cos x$ and $\sin x$

(b) Hence, find $\int \sin x \cos x \, dx$

WORKING AT B/C

(1) (a) Prove that $(\cos x + \sin x)^2 \equiv 1 + \sin 2x$

(b) Hence, find $\int_0^{\frac{\pi}{6}} (\cos x + \sin x)^2$ giving your answer in exact form.

(2) (a) Write $\cot^2 x$ in terms of $\operatorname{cosec}^2 x$

(b) Hence, find $\int 3\cot^2 x \;\, dx$

(3) (a) Show that $(\cos x + \tan x)^2$ can be written as:

(i) $\cos^2 x + 2\sin x + \tan^2 x$

(ii) $\frac{1}{2}\cos 2x + 2\sin x + \sec^2 x - \frac{1}{2}$

(b) Using your answer to part (ii), find

$$\int (\cos x + \tan x)^2 \;\, \mathrm{dx}$$

WORKING AT A*/A

(1) Show that:

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} (1 - \sin 2x)^2 = \frac{1}{16}\left(3\pi + 2 - 8\sqrt{2}\right)$$

(2) Evaluate $\int_0^{\pi} \left(4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) + \left(2\cos^2\frac{\theta}{2} - 1\right) \, d\theta$

(3) (a) Show that $\frac{1 + \cos x - \cos^2 x}{\sin x} \equiv \cot x + \sin x$

(b) Hence, find $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1 + \cos x - \cos^2 x}{\sin x} \; dx$ giving your answer in the form $a + \ln b$

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(63) Integration by Inspection (Reverse Chain Rule)

WORKING AT D/E

(1) Find the integral of each of the following 'by inspection'

(a) $\int 2e^{2x}\,dx$ (b) $\int \frac{2x}{x^2+1}\,dx$ (c) $\int 4(2x+1)^5\,dx$

(d) $\int -3\sin x(\cos x+1)^2\,dx$ (e) $\int 8\sin 2x\,\,dx$

(2) Explain why $\int(\cos x)\sin^5 x\,dx = \frac{1}{6}\sin^6 x + c$

(3) Show each of the following:

(a) $\int_1^2 \frac{4x-1}{2x^2-x}\,dx = \ln 6$

(b) $\int_0^3 4xe^{x^2}\,dx = 2e^9 - 2$

WORKING AT B/C

(1) (a) Using the formula book, write down $\frac{d}{dx}\sec x$.

(b) Hence, find $\int \tan x \sec^6 x\,dx$

(2) Evaluate $\int_0^1 2x(2x^2-3)^5\,dx$ showing each step of your workings.

(3) Show that $\int_0^{\frac{\pi}{2}} -\sin x\cos^3 x\,dx = -0.25$

WORKING AT A*/A

(1) Find $\int \cot 2x\,\mathrm{cosec}^6\,2x\,\,dx$

(2) Show that $\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx = \ln\sqrt{\frac{e^2+1}{2}}$

(3) Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \mathrm{cosec}^2 x\cot^3 x\,dx = 2$

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(64) Integration by Substitution

WORKING AT D/E

(1) (a) Given that $u = 3x + 2$, find an expression for $\frac{du}{dx}$ in terms of x.

(b) Using your answer to part (a) show that $\int e^{3x+2}\,dx$ can be written as $\int \frac{1}{3}e^u\,du$

(c) Find $\int \frac{1}{3}e^u\,du$ in terms of u.

(d) Hence, find $\int e^{3x+2}\,dx$ in terms of x

(2) (a) Using the substitution $u = \sin x$ show that $\int 2\cos x\, e^{sinx}\,dx$ can be written as $\int 2e^u\,du$

(b) Use your answer to part (a) to find $\int 2\cos x\, e^{sinx}\,dx$ in terms of x

(3) (a) Using the substitution $u = x - 6$, show that $\int x\sqrt{x-6}\,dx$ can be written as $\int (u+6)u^{\frac{1}{2}}\,du$

(b) Hence, show that

$$\int x\sqrt{x-6}\,dx = \frac{2}{5}(x-6)^{\frac{5}{2}} + 4(x-6)^{\frac{3}{2}} + c$$

WORKING AT B/C

(1) (a) Using the substitution $u = \sin x$, show that $\int \cos x \sin^7 x\,dx = \frac{1}{8}\sin^8 x + c$

(b) Using the substitution $u = 4x - 3$ show that $\int 16x\sqrt[3]{4x-3}\,dx = \frac{3}{7}(4x-3)^{\frac{7}{3}} + \frac{9}{4}(4x-3)^{\frac{4}{3}} + c$

(2) Using thew substation $u = \tan x$, show that

$$\int_0^{\frac{\pi}{4}} \sec^2 x e^{\tan x}\,dx = e - 1$$

(3) Using the substitution $u = 1 + \sin 2x$, show that

$$\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1+\sin 2x}\,dx = \ln\sqrt{2}$$

WORKING AT A*/A

(1) Prove that $\int \frac{1}{(1-x^2)^{\frac{1}{2}}}\,dx = \arcsin x + c$

(2) Use the substitution $u^3 = 6x + 1$ to show that $\int \frac{3x}{\sqrt[3]{6x+1}}\,dx = \frac{3}{40}(4x-1)(6x+1)^{\frac{2}{3}} + c$

(3) Use the substitution $u^2 = e^x + 1$ to find $\int_0^{\ln 3} \frac{e^{3x}}{e^x+1}\,dx$ giving your answer in the form $a + \ln b$

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(65) Integration by Parts

WORKING AT D/E

(1) **Use the formula** book and LATE to show that:

(a) $\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x \, dx$

(b) Hence, find $\int x \ln x \, dx$

(2) Use integration by parts to show that

$$\int xe^x dx = xe^x - e^x + c$$

(3) **Use the formula book** to show each of the following integrals:

(a) $\int x \cos x \, dx = x \sin x + \cos x + c$

(b) $\int e^x \ln x \, dx = e^x \ln x - \int \frac{e^x}{x} dx + c$

WORKING AT B/C

(1) (a) Show that $\int_1^e x^2 \ln x \, dx = \frac{1}{9}(2e^2 + 1)$

(b) Show that $\int_0^{\frac{\pi}{2}} x \sin x \, dx = 1$

(2) **Use the formula** book to find $\int 2x \sec^2 \, x \, dx$

(3) Find each of the following. Give each answer in exact form showing full workings:

(a) $\int_0^2 xe^x dx$

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2x \cos x \, dx$

WORKING AT A*/A

(1) Show that $\int_0^1 x^2 e^{2x} dx = \frac{e^2-1}{4}$

(2) Evaluate $\int_1^{2e} \sqrt{x} \ln 2x \, dx$ giving your answer in exact form.

(3) Show that $\int e^x \sin x \, dx = \frac{e^x}{2}(\sin x - \cos x) + c$

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(66) Integration using Partial Fractions

WORKING AT D/E

(1) (a) Using partial fractions, show that $\frac{5+4x}{(1+x)(2+x)}$ can be written as $\frac{1}{1+x}+\frac{3}{2+x}$

(b) Hence, show that $\int \frac{5+4x}{(1+x)(2+x)}dx$ can be written as the sum of natural logarithms.

(2) (a) Express $\frac{x+13}{(1-x)(6+x)}$ in the form $\frac{A}{(1-x)}+\frac{B}{(6+x)}$

(b) Hence, using integration and the laws of logarithms, show that

$$\int \frac{x+13}{(1-x)(6+x)}dx = \ln\frac{|6+x|}{(1-x)^2}+c$$

WORKING AT B/C

(1) Doris wants to express $\frac{-1}{(1+2x)(1+x)^2}$ in partial fractions. She writes

$$\frac{-1}{(1+2x)(1+x)^2} \equiv \frac{A}{(1+2x)}+\frac{B}{(1+x)^2}$$

(a) Explain the error that she has made.

(b) Show, using partial fractions that

$$\frac{-1}{(1+2x)(1+x)^2} \equiv \frac{2}{(1+x)}+\frac{1}{(1+x)^2}-\frac{4}{(1+2x)}$$

(c) Hence, show that

$$\int \frac{-1}{(1+2x)(1+x)^2}dx = \ln\left(\frac{1+x}{1+2x}\right)^2 - \frac{1}{1+x}+c$$

(2) (a) Factorise $1-16x^2$

(b) Hence, express $\frac{5+4x}{1-16x^2}$ in partial fractions

(c) Show that $\int_0^{0.1}\left(\frac{5+4x}{1-16x^2}\right)dx = \ln\left(\frac{\sqrt{1.4}}{\sqrt[4]{0.216}}\right)$

WORKING AT A*/A

(1) (a) Show that $\frac{11+4x-2x^2}{6+x-x^2}$ can be written in the form

$$A+\frac{B}{3-x}-\frac{C}{2+x}$$

(b) Hence, show that

$\int_0^2 \frac{11+4x-2x^2}{6+x-x^2}dx = p+\ln q$ where p and q are rational constants to be found.

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A LEVEL MATHS

(67) Integration to Find Areas

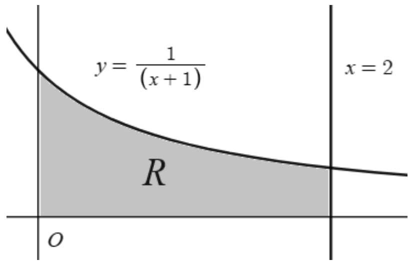
WORKING AT D/E

(1) (a) Sketch the graph of $y = \cos(x)$ for $0 \leq x \leq 2\pi$.

(b) Write down the coordinates of the points A and B where the graph crosses the x axis.

(c) Use integration to show that the area trapped between the curve and the x axis between the points A and B is 2 units

(2) The diagram below shows part of the curve with equation $y = \frac{1}{x+1}$ and the line with the equation $x = 2$.

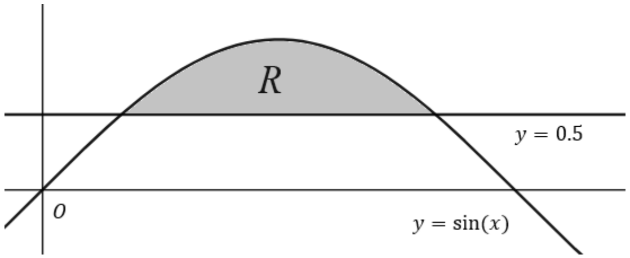


The region R is the shaded region enclosed the curve, the line and the positive x and y axis.

Show that the area of R is $\ln A$ where A is an integer to be found.

WORKING AT B/C

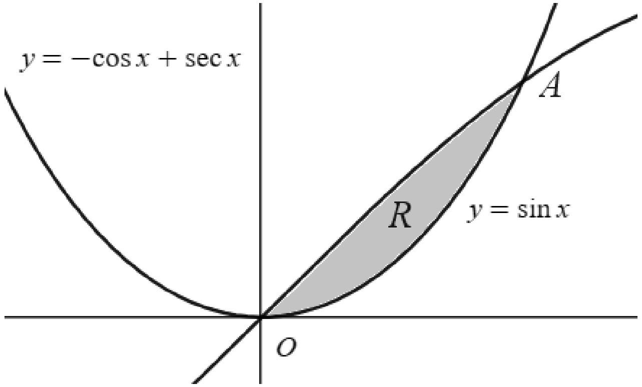
(1) The diagram below shows part of the curve with equation $y = \sin(x)$ and part of the line with equation $y = 0.5$



The shaded region R is the area enclosed between the line and the curve.

Show that the area of R is $\sqrt{3} - \frac{\pi}{3}$

(2) The diagram below shows part of the graphs of $y = -\cos x + \sec x$ and $y = \sin x$.



The graphs intersect at the point A

(a) Find the coordinates of A

(b) Using the formula book, find the area of the region R. Give your answer in exact form.

WORKING AT A*/A

(1) (a) On the same set of axes sketch the graphs of $y = \cos(2x)$ and $y = -\sin(x)$ for $-\pi \leq x \leq \pi$

(b) The graphs of $y = \cos(2x)$ and $y = -\sin(x)$ intersect at the point A and B in the interval $-\pi \leq x \leq \pi$. Find the x coordinates of the A and B giving your answers in exact form.

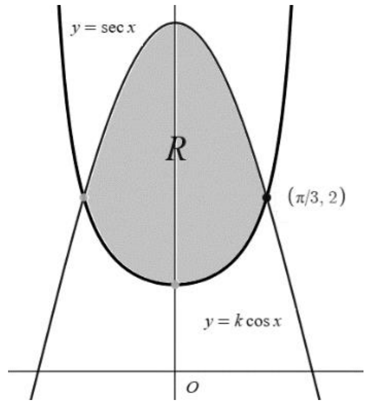
(c) Show that the area trapped between the two curves between A and B can be written in the form $p\sqrt{3} + q$ where p and q are rational fractions.

(2) The curve C has parametric equations:

$$x = e^{t+1}, \qquad y = t^2 - 4, \qquad t \in R$$

The curve crosses the x axis at A and B. Find the exact area trapped between the curve and the positive x axis between A and B.

(3) The diagram below shows part of the curves with equations $y = k\cos(x)$ and $y = \sec(x)$.



Given that the curves intersect at the point with coordinates $\left(\frac{\pi}{3}, 2\right)$, show that exact area of the region R is $4\sqrt{3} - \ln\left(2 + \sqrt{3}\right)^2$

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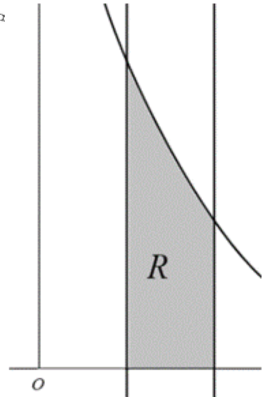
(68) Integration using the Trapezium Rule

WORKING AT D/E

(1) The diagram below shows part of the curve with equation $y = \frac{3}{x^{\frac{1}{2}}} + \cos(x)$ where x is measured in radians. The diagram also shows the lines $x = 1$ and $x = 2$. The region R is the area between the curve, the lines $x = 1, x = 2$ and the positive x axis.



(a) Complete the table below for $y = \frac{3}{x^{\frac{1}{2}}} + \cos(x)$ giving each answer to 4 significant figures.

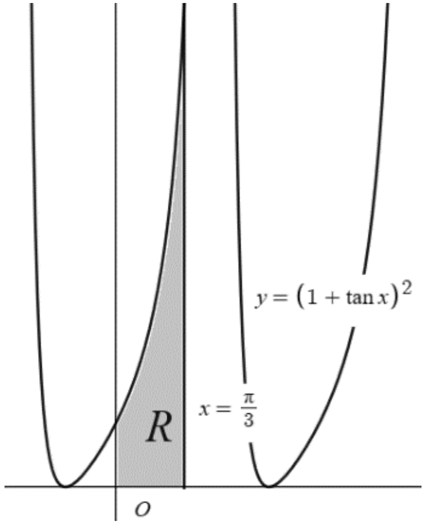
x	1	1.25	1.5	1.75	2
y					

(b) Use the trapezium rule with 4 equal strips to estimate the area of the region R to 4 S.F

(c) Explain how you could find a more accurate value to your answer in part (b)

WORKING AT B/C

(1) The diagram below shows part of the curve with equation $y = (1 + \tan(x))^2$ where x is measured in radians. The diagram also shows the line $x = \frac{\pi}{3}$ .The region R is the area between the curve, the line $x = \frac{\pi}{3}$, the positive x axis and the positive y axis.



(a) Complete the table for $y = (1 + \tan(x))^2$ giving each answer to 4 significant figures where appropriate.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y					

(b) Use the trapezium rule with 4 equal strips to estimate the area of the region R

(c) Explain why the estimation in part (a) is an overestimate.

(d) Show that $(1 + \tan(x))^2 \equiv 2\tan(x) + \sec^2(x)$

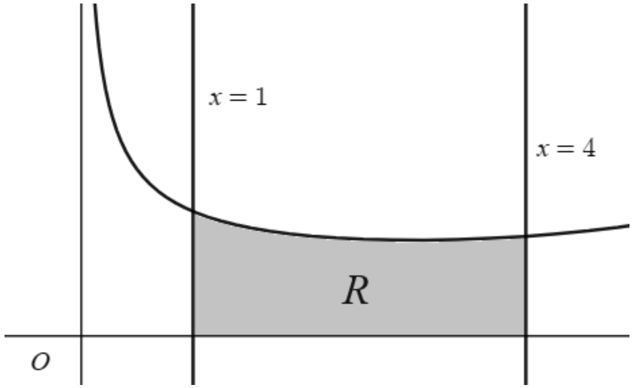
(e) Hence, use integration to show the exact area of R is $\sqrt{3} + \ln 4$

(f) Find the % error in your answer to part (b)

WORKING AT A*/A

(1) The diagram below shows part of the curve with equation $y = \frac{-9}{x^{0.5}(x-9)}$

The diagram also shows the lines $x = 1$ and $x = 4$. The region R is the area between the curve, the lines $x = 1, x = 4$ and the positive x axis.



(a) Complete the table below for $y = \frac{-9}{x^{0.5}(x-9)}$ giving each answer to 4 significant figures.

x	1	1.75	2.5	3.25	4
y					

(b) Use the trapezium rule with 4 equal strips to estimate the area of the region R.

(c) Use the substitution $u = x^{0.5}$ to find the exact value of $\int_1^4 \frac{-9}{x^{0.5}(x-9)} dx$

(d) Find the % error in your answer to part (b).

(e) Without any further calculations, explain whether the trapezium rule would give an overestimate or underestimate for $y = \frac{9}{x^{0.5}(x-9)}$ for the same interval.

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(69) Solving Differential Equations

WORKING AT D/E

(1) Find a general solution to each differential equation:

(a) $\frac{dy}{dx} = \frac{x}{e^y}$ (b) $\frac{dy}{dx} = \frac{\cos 2x}{\sin y}$ (c) $\frac{dy}{dx} = y\sec^2 x$

(2) Give that when $x = 1, y = \frac{\pi}{2}$ show that the particular solution to the differential equation

$x\frac{dy}{dx} = \sec y$ can be written as $x = e^{\sin(y)-1}$

(3) Find the particular solution to the differential equation $\frac{dy}{dx} = y^2 e^{2x}$ given the equation satisfies the boundary conditions $y = 1$ at $x = 0$. Give your answer in the form $y = f(x)$

WORKING AT B/C

(1) (a) Express $\frac{14-2x}{(1+x)(3-x)}$ in partial fractions.

(b) Hence, find a general solution to the differential equation $(1+x)(3-x)\frac{dy}{dx} = \frac{2(7-x)}{y}$ in the form $y^2 = f(x)$

(2) The differential equation $\cos^2(x)\frac{dy}{dx} = y$ has boundary condition $y = 1$ at $x = \frac{\pi}{4}$.

Show that the origin $(0, \frac{1}{e})$ also satisfies the equation.

(3) A differential equation is such that $\frac{dy}{dx} = x^2$

On the same set of axes, draw 3 different particular solutions to the differential equation.

WORKING AT A*/A

(1) Show that the particular solution to the differential equation $e^{y-x}\frac{dy}{dx} = -x$ with boundary conditions $y = 0$ at $x = 0$ can be written as $y = x + \ln(1-x)$

(2) Find a general solution to the differential equation $\frac{dy}{dx} = (\ln x)\operatorname{cosec}^2(y)$

(3) Given that one particular solution to the differential equation $\frac{dy}{dx} = 3^x e^{-3y}$ passes through the origin, prove that $\ln 3(e^{3y} - 1) + 3 = 3^{x+1}$

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(70) The Applications of Differential Equations

WORKING AT D/E

(1) The rate at which the number of rats in a colony is increasing can be modelled by the differential equation $\frac{dN}{dt} = 0.1N$ where N is the number of rats (in thousands) and t is the time in days.

(a) Show that the general solution to the differential equation is $N = Ae^{0.1t}$ where A is a constant.

(b) Given that there were initially 6000 rats, write down the value of A.

(c) Find the population of the rats after a week.

(d) Sketch the graph of the population of the rats over time.

(e) State a limitation of the model.

WORKING AT B/C

(1) The rate at which a mould patch is shrinking in a shower is proportional the amount already present.

(a) Show that the amount of mould in the shower satisfies the equation $M = Ae^{kt}$ where M is the mass of the mould, t is the time in days and A and k are constants.

(b) Given that there was initially $40g$ of mould and that after 20 days there was $8g$ of mould, find the value of A and k. Give the value of k to 3 significant figures.

(c) Find when the mass of the mould will fall below $2g$.

(d) Sketch the graph of the mass of the mould over time.

WORKING AT A*/A

(1) Juice is being poured into a cooler at a steady rate of $20cm^3s^{-1}$. The cooler is leaking at a rate of one tenth of the current volume (V) of juice in the cooler .

(a) Show that V satisfies the equation

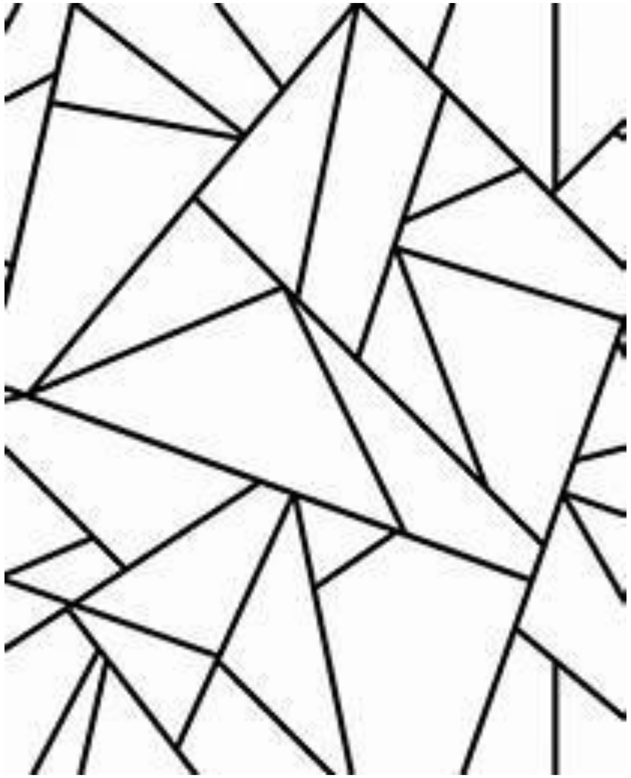
$$V = 200 + Ae^{-0.1t}$$

Where A is a constant and t is the time in seconds.

(b) Given that the initial volume in the cooler is $280cm^3$, find the time taken for volume to reach $250cm^3$

(c) Doris needs to ensure she always has enough in the cooler for one glass of juice. Given the glass holds $180ml$, explain why Doris will always have enough juice in the container for at least one glass.

Vectors



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(71) 3D Coordinates

WORKING AT D/E

(1) Show that the distance from the origin to the point $Q(-2,7,5)$ is $\sqrt{78}$

(2) Find the distance between the points $P(-4,7,-2)$ and $Q(3,5,0)$

(3) Find the coordinates of any point that is a distance of 8 units from O

WORKING AT B/C

(1) The distance $PQ = \sqrt{105}$. Given that $P(-3,2,0)$ and $Q(1,-6,q)$ find the possible values of q.

(2) Find the coordinates of the point on the positive z axis that is a distance of $5\sqrt{5}$ from the point $P(10,3,1)$

WORKING AT A*/A

(1) In the square $ABCD$, $A(-1,4,7)$ and $C(9,10,-1)$ Find the perimeter of $ABCD$.

(2) Points $P(4,0,0)$, $Q(0,4,0)$ and $R(0,0,r)$ form an equilateral triangle.
(a) Write down the possible values of r
(b) Find the exact area of the triangle PQR.

(3) Point $P(-p,-p,-p)$ where p is a positive constant is $3\sqrt{3}$ from O. Find p.

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(72) Vectors in 3D

WORKING AT D/E

(1) Point $A(4,-2,6)$

(a) Find the position vector $\overrightarrow{OA}$ using **i j k** notation.

(b) Find the position vector $\overrightarrow{OA}$ as a column vector.

(c) Find $|\overrightarrow{OA}|$

(2) Point $A(1,0,-12)$ and $B(3,-2,8)$

(a) Find the position vector $\overrightarrow{OA}$ using **i j k** notation.

(b) Find the position vector $\overrightarrow{OB}$ using **i j k** notation.

(c) Find the direction vector $\overrightarrow{AB}$ using **i j k** notation.

(d) Find $|\overrightarrow{AB}|$

(3) $\overrightarrow{OA} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

Find $|\overrightarrow{AB}|$

WORKING AT B/C

(1) $\overrightarrow{OA} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OB} = \mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$

(a) Find $|\overrightarrow{AB}|$

(b) Hence find the perimeter of triangle OAB in exact form.

(2) (a) Find the angle the vector $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ makes with each of the coordinate axes.

(b) Find a unit vector in the direction of $\mathbf{a}$

(3) Points $P(3,2,-4), Q(1,4,3)$ and $R(-1,-5,4)$

(a) Find the exact value of $\cos(PQR)$

(b) Hence, show that the value of $\sin(PQR) = 0.994989$ correct to 6SF.

WORKING AT A*/A

(1) The point A is such that $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$

Find the perpendicular distance of A from the positive x axis.

(2) The vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular.

$\overrightarrow{OA} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{j} - \mathbf{k}$

Find the exact value of $\cos(OAB)$.

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(73) Vector Geometry

WORKING AT D/E

(1) Given that

$(p - 1)\mathbf{i} + (q+2)\mathbf{j} + r\mathbf{k} = -8\mathbf{i} + 22\mathbf{j}$

find the values of p, q and r

(2) $\overrightarrow{OA} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\overrightarrow{OD} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$.

(a) Find the vectors:

(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{BC}$ (iii) $\overrightarrow{DC}$ (iv) $\overrightarrow{AD}$

(b) Find (i) $|\overrightarrow{AB}|$ (ii) $|\overrightarrow{BC}|$ (iii) $|\overrightarrow{DC}|$ (iv) $|\overrightarrow{AD}|$

(c) Hence, explain why $ABCD$ is a parallelogram.

WORKING AT B/C

(1) $ABCD$ is a trapezium.

$\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

$\overrightarrow{OB} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\overrightarrow{OC} = -\mathbf{i} + 7\mathbf{k}$

$\overrightarrow{OD} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$

Given that $\overrightarrow{AD} = 2\overrightarrow{BC}$

(a) Find the values of p, q and r

(b) Hence, find the exact lengths of the parallel sides in the trapezium.

WORKING AT A*/A

(1) $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}$

Explain why $0 < \cos(ABC) < 1$

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A LEVEL MATHS

(74) Vectors in Mechanics

WORKING AT D/E

(1) Two forces act on a particle of mass $8kg$

$F_1 = (5\mathbf{i} - 2\mathbf{j} + \mathbf{k})N$ and $F_2 = (2\mathbf{i} - \mathbf{j} + 4\mathbf{k})N$

(a) Explain why the resultant force R is such that $R = (7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})N$

(b) Find the magnitude of R

(c) Using Newtons 2nd Law, explain why the particle will accelerate at $1.14ms^{-2}$ correct to 3SF

(d) A 3rd force now acts on the particle.

$F_3 = (-7\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})N$

Explain why the particle would be in equilibrium if all 3 forces acting on it.

WORKING AT B/C

(1) A particle of mass $4kg$ is acted upon by 3 forces:

$$F_1 = \begin{pmatrix} -1 \\ 0 \\ 12 \end{pmatrix} N \quad F_2 = \begin{pmatrix} 10 \\ -3 \\ 1 \end{pmatrix} N \quad F_3 = \begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix} N$$

(a) Find the acceleration a of the particle in the form $\mathrm{p}\mathbf{i} + \mathrm{q}\mathbf{j} + \mathrm{r}\mathbf{k}$

(b) Hence find $|a|$

(c) The particle is initially at rest. Find the distance covered by the particle in the first 5 seconds after the forces have acted upon it.

(d) Find the angle the acceleration makes with the vector i.

WORKING AT A*/A

(1) Cyril and Doris are trying to keep a ping pong ball in the air using two hairdryers. The ping pong ball has mass 20g. The two hairdryers produce forces of $H_1 = (\mathrm{p}\mathbf{i} - 5\mathbf{j} + \mathrm{r}\mathbf{k})N$ and $H_2 = (2\mathbf{i} + \mathrm{q}\,\mathbf{j} + (\mathrm{r}-1)\mathbf{k})N$ where p, q and r are constants. The hairdryers are applied simultaneously to keep the ping pong ball in the air.

As soon as the hairdryers are turned on the ping pong ball travels directly upwards from a fixed starting point with acceleration $6\mathbf{k}\, ms^{-2}$

Find the values of p, q and r.