Complex Number Numbers – CP2 Edexcel – Steve Blades – www.m4ths.com

(1) Show that $\left(3 - i\sqrt{3}\right)^{12}$ is a real number that can be written in the from a^n.

(2) (a) Show that $\sin^4\theta$ can be written in the form $A\cos 4\theta + B\cos 2\theta + C$ where A, B and C are constants to be found.
(b) Hence, show that:

$$32\int_0^{\frac{\pi}{4}} \sin^4\theta = 3\pi - 8$$

Answers

(3) $z = \frac{\sqrt{3}+i}{\sqrt{2}(1+i)}$.
(a) Write z in the mod/arg form.
(b) Plot z on an Argand diagram.
(c) Find the coordinates of z in Cartesian form.
(d) The point P has the coordinates found in part (c) of the question. P is one of the vertices of an equilateral triangle. Find the coordinates other two vertices of the equilateral triangle.

(4) Solve the equation $z^5 - 4 + 4i = 0$ giving your answers in exponential form.

(5) z and w are 2 different complex numbers such that:

$$|z| = 4$$
$$\arg\left(\frac{w}{z}\right) = \frac{\pi}{6}$$
$$w = -4 + 4\sqrt{3}i$$

Show that z is purely imaginary and plot it on an Argand diagram.

(6) Given that $z = e^{\frac{\pi i}{4n}}$, show that:

$$1 + z + z^2 + z^3 + \cdots + z^{8n-1} = 0$$

(7) The coordinates of one vertex a square are $\left(1, -\sqrt{3}\right)$.
(a) Find the coordinates of the other 3 vertices.
(b) Find the area of the square.
(c) The midpoints to the sides of the square are the $4th$ roots of another complex number w.
Find a possible expression for w in the form $re^{i\theta}$.

(8) Find the $6th$ roots of unity giving your answers in the form $x + iy$.

(9) Two convergent infinite series C and S are given as:

$$C = 1 + \frac{1}{5}\cos\theta + \frac{1}{25}\cos 2\theta + \frac{1}{125}\cos 3\theta + \cdots$$
$$S = \frac{1}{5}\sin\theta + \frac{1}{25}\sin 2\theta + \frac{1}{125}\sin 3\theta + \cdots$$

(a) Show that:

$$C + iS = \frac{5}{5 - e^{i\vartheta}}$$

(b) Hence show that:

$$C = \frac{25 - 5\cos\theta}{26 - 10\cos\theta}$$

(c) Find a similar expression for S.

(10) (a) Show that $\cos 5\theta = \cos\theta(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta)$
(b) Hence, solve the equation $\cos\theta(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta) = \sec 5\theta,\ 0 \le \vartheta \le \frac{\pi}{2}$
(c) Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Answers Complex Numbers www.m4ths.com

① Modus $= \sqrt{12}$ or $12^{\frac{1}{2}}$

Argument $= -\frac{\pi}{3}$

$\therefore \left(3 - i\sqrt{3}\right)^{12} = \left(\sqrt{12}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right)^{12}$

$= \left(\left(12^{\frac{1}{2}}\right)^{6}\right)\left[\cos\left(-4\pi\right) + i\sin\left(-4\pi\right)\right]$

$= 12^{6}\left[\cos(0) + i\sin(0)\right]$

$= 12^{6}\left(1 + 0i\right)$

$\underline{\underline{12^{6}}}$

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②a $z - \frac{1}{z} = 2i\sin\theta$ and $z + \frac{1}{z} = 2\cos\theta$

$\therefore \left(z - \frac{1}{z}\right)^{4} = \left(2i\sin\theta\right)^{4}$

$16\sin^{4}\theta = z^{4} + 4\left(z^{3}\right)\left(-\frac{1}{z}\right) + 6\left(z^{2}\right)\left(\frac{1}{z^{2}}\right) + 4\left(z\right)\left(-\frac{1}{z^{3}}\right) + \left(\frac{1}{z^{4}}\right)$

$= \left(z^{4} + \frac{1}{z^{4}}\right) - 4\left(z^{2} + \frac{1}{z^{2}}\right) + 6(1)$

$= 2\cos 4\theta - 4(2\cos 2\theta) + 6$

$\therefore \sin^{4}\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$ $\quad A = \frac{1}{8}, B = -\frac{1}{4}, C = \frac{3}{8}$

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②b $32\int_{0}^{\pi/4} \sin^{4}\theta = \int_{0}^{\pi/4} 32\left[\frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}\right] d\theta$

$= \int_{0}^{\pi/4} 4\cos 4\theta - 16\cos 2\theta + 12 \, d\theta$

$= \left[\sin 4\theta - 8\sin 2\theta + 12\theta\right]_{0}^{\frac{\pi}{4}}$

$= \left(0 - 8(1) + 3\pi\right) - \left(0 + 0 + 0\right)$

$= 3\pi - 8$ ✓

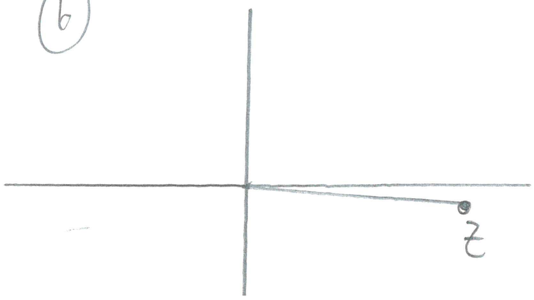
(3) $z = \dfrac{\sqrt{3} + i}{\sqrt{2} + \sqrt{2}i}$

(a) $= \dfrac{\cancel{2}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)}{\cancel{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)}$

$= \cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)$

$= \cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)$

(b)



(c) $\left(\dfrac{\sqrt{6}+\sqrt{2}}{4}, \dfrac{\sqrt{6}-\sqrt{2}}{4}\right)$

(d) $z = e^{-\frac{\pi}{12}i}$ $\quad w = e^{\frac{2\pi k}{3}i}$

when $k=1$

$zw = e^{-\frac{\pi}{12}i} \times e^{\frac{2\pi}{3}i}$

$= e^{\frac{7\pi}{12}i}$

$\therefore \left(\dfrac{-\sqrt{6}+\sqrt{2}}{4}, \dfrac{\sqrt{6}+\sqrt{2}}{4}\right)$ ✓

When $k=2$ (or -1)

$zw = e^{-\frac{\pi}{12}i} \times e^{\frac{4\pi}{3}i}$

$= e^{\frac{5\pi}{4}i}$

$\therefore \left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$ ✓

(4) $z^5 - 4 + 4i = 0$

$z^5 = 4 - 4i$

$(r(\cos\theta + i\sin\theta))^5 = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4} + 2\pi k\right) + i\sin\left(-\frac{\pi}{4} + 2\pi k\right)\right)$

$r^5 = 4\sqrt{2} \Rightarrow r = \sqrt{2}$

$5\theta = -\frac{\pi}{4} + 2\pi k \Rightarrow \theta = -\frac{\pi}{20} + \frac{2\pi k}{5}$

$z = \sqrt{2}e^{-\frac{\pi}{20} + \frac{2\pi k}{5}}$

When $k=0$: $z_1 = \sqrt{2}e^{-\frac{\pi}{20}i}$

when $k=1$: $z_2 = \sqrt{2}e^{7/20\,\pi i}$

when $k=2$: $z_3 = \sqrt{2}e^{\frac{3\pi}{4}i}$

when $k=3$: $z_4 = \sqrt{2}e^{\frac{-17}{20}\pi i}$

when $k=-1$: $z_5 = \sqrt{2}e^{-\frac{9}{20}\pi i}$

(5) $w = -4 + 4\sqrt{3}i$

(a) Mod $= 8$ $\quad$ arg $= -\frac{\pi}{3}$

$\arg\left(\frac{w}{z}\right) \equiv \arg w - \arg z$

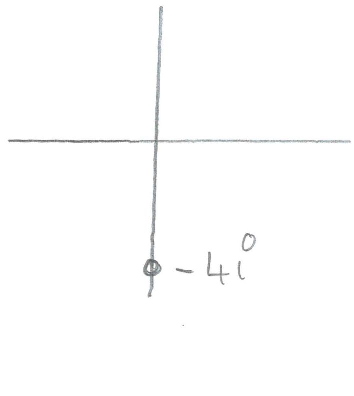
$\therefore \frac{\pi}{6} = -\frac{\pi}{3} - \arg z$

$\arg z = -\frac{\pi}{2}$

$z = 4e^{-\frac{\pi}{2}i}$

$z = -4i$

(b)



6 Geometric Series

$$\text{Using } S = \frac{a(r^n - 1)}{r - 1}$$

$a = 1$
$r = e^{\frac{\pi}{4n} i}$
$S =$
$n = 8n$

$$S = \frac{1\left(\left(e^{\frac{\pi}{4n} i}\right)^{8n} - 1\right)}{e^{\frac{\pi}{4n} i} - 1} \Rightarrow \frac{e^{2\pi i} - 1}{e^{\frac{\pi i}{4n}} - 1} \Rightarrow \frac{(1 + 0i) - 1}{e^{\frac{\pi i}{4n}} - 1} \Rightarrow \frac{0}{e^{\frac{\pi i}{4n}} - 1} \Rightarrow 0$$

✓

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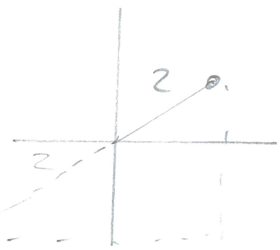
7 $z = 2e^{-\frac{\pi}{3} i}$ $\quad w = e^{\frac{\pi}{2} k i}$

a

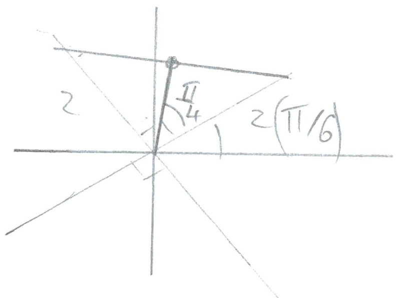
when $k=1$	when $k=2$	when $k=3$
$zw = 2e^{\frac{\pi}{6} i}$	$zw = 2e^{\frac{2\pi}{3}}$	$zw = 2e^{-\frac{5\pi}{6}}$
Coordinates	Coordinates	Coordinates
$\sqrt{3}, 1$	$-1, \sqrt{3}$	$-\sqrt{3}, -1$

b



$x^2 + x^2 = 4^2$
$2x^2 = 16$
$x^2 = 8$ ✓

c



$\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$

$\text{Mod} = 2\cos\frac{\pi}{4}$
$= \sqrt{2}$

$\therefore w^{\frac{1}{4}} = \sqrt{2} e^{\frac{5\pi}{12} i}$

$\therefore w = 4e^{\frac{5\pi}{3} i}$
$= 4e^{-\frac{\pi}{3} i}$ o.e

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8 $z = e^{\frac{\pi}{3} k i}$

$k=0$	$k=1$	$k=2$	$k=3$	$k=-1$	$k=-2$
$\therefore z = e^0 = 1$	$z = e^{\frac{\pi}{3} i}$	$z = e^{\frac{2\pi}{3} i}$	$z = e^{\pi i}$	$z = e^{-\frac{\pi}{3} i}$	$z = e^{-\frac{2\pi}{3} i}$
	$z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$	$z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$	$z = -1$	$z = \frac{1}{2} - \frac{\sqrt{3}}{2} i$	$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}$

(a) $C + iS = 1 + \frac{1}{5}\cos\theta + \frac{1}{5}i\sin\theta + \frac{1}{25}\cos 2\theta + \frac{1}{25}i\sin 2\theta + \dots$

(a) $= 1 + \frac{1}{5}(\cos\theta + i\sin\theta) + \frac{1}{25}(\cos 2\theta + i\sin 2\theta) + \dots$

$= 1 + \frac{1}{5}z + \frac{1}{25}z^2 + \frac{1}{125}z^3$

Geometric Series

$a = 1$

$r = \frac{1}{5}z$ or $\frac{1}{5}e^{i\theta}$

$n = \infty$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1-\frac{1}{5}e^{i\theta}}$$

$$= \frac{5}{5-e^{i\theta}}$$ ✓

(b) $\frac{5}{5-(\cos\theta + i\sin\theta)} \equiv \frac{5}{(5-\cos\theta) + i\sin\theta}$

Realizing

$$\frac{5[(5-\cos\theta) - i\sin\theta]}{[(5-\cos\theta) + i\sin\theta][(5-\cos\theta) - i\sin\theta]}$$

$$\frac{5(5-\cos\theta) - 5i\sin\theta}{25 - 10\cos\theta + \cos^2\theta - \cancel{i\sin\theta(5-\cos\theta)} + \cancel{i\sin\theta(5-\cos\theta)} + \sin^2\theta}$$

$$\frac{5(5-\cos\theta) - 5i\sin\theta}{25 - 10\cos\theta + 1}$$

Real Parts

$$\frac{25 - 5\cos\theta}{26 - 10\cos\theta}$$ ✓

(c) Im parts $\frac{-5\sin\theta}{26 - 10\cos\theta}$ ✓

(10a) $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$\cos 5\theta + i\sin 5\theta \equiv \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(-\sin^2\theta)$
$+ 10\cos^2\theta(-i\sin^3\theta) + 5\cos\theta(\sin^4\theta)$

Equating Real Parts

$\cos 5\theta \equiv \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$
$\equiv \cos\theta[\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta]$ ✓

(b) $\cos 5\theta = \sec 5\theta$

$\cos 5\theta = \frac{1}{\cos 5\theta}$

$\cos^2 5\theta = 1$

$\cos 5\theta = \pm 1$

$\cos 5\theta = 1$

$\therefore 5\theta = 0 \pm 2\pi n$

$5\theta = 2\pi \pm 2\pi n$

$\therefore \theta = 0 \pm \frac{2}{5}\pi n$

$\therefore \theta = 0, \frac{2}{5}\pi$

$\cos 5\theta = -1$

$\therefore 5\theta = \pi \pm 2\pi n$

$\theta = \frac{\pi}{5} \pm \frac{2\pi n}{5}$

$\therefore \theta = \frac{\pi}{5}$

$\underline{\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}}$

(c) B when $n=1$

LHS $(\cos\theta + i\sin\theta)^1$ RHS $(\cos\theta + i\sin\theta)$ ✓

True for $n=1$, LHS=RS

(A) If true for $n=1$, assume true for $n=k$, $k \in \mathbb{Z}^+$
such that $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$.

(I) If true for $n=k$, assume true for $n=k+1$ such that
$(\cos\theta + i\sin\theta)^{k+1} \equiv \cos(k+1)\theta + i\sin(k+1)\theta$

LHS

$$(\cos\theta + i\sin\theta)^{k+1} \equiv (\cos\theta + i\sin\theta)^{k} \times (\cos\theta + i\sin\theta)^{1}$$

$$\equiv (\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta)$$

$$\equiv \cos(k\theta)\cos\theta + (i\sin\theta)\cos k\theta + (i\sin(k\theta))\cos\theta - i\sin(k\theta)(\sin\theta)$$

$$\equiv \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$ by the addition formula

$$\equiv \cos(k+1)\theta + i\sin(k+1)\theta \checkmark$$

$\therefore$ True when $n = k+1$

(c) If De Moivres is true for $n = k$, then shown true for $n = k+1$. As it's true for $n = 1$, it's true for all $n \in \mathbb{Z}^+$ by mathematical induction.