Methods in Differential Equations www.m4ths.com

① (a) I.F $e^{\int 3dx} = e^{3x}$. Multiplying through gives:

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = xe^{2x}$$

$$\therefore ye^{3x} = \int xe^{2x}\,dx$$

I.B.P

u	$\frac{du}{dx}$
x	1
v	$\frac{dv}{dx}$
$\frac{1}{2}e^{2x}$	e^{2x}

$$ye^{3x} = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$ye^{3x} = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

$$ye^{3x} = \frac{1}{4}e^{2x}(2x-1) + c$$

$$y = \frac{1}{4}e^{-x}(2x-1) + ce^{-3x}$$

(b) $3 = \frac{1}{4}(1)(0-1) + c(1)$

$\frac{13}{4} = c$

$$\therefore y = \frac{1}{4}e^{-x}(2x-1) + \frac{13}{4}e^{-3x}$$

(c) as $x \to \infty$ $y \to 0$

② Aux Eq $m^2 + 4 = 0$

$\therefore m = \pm 2i$

C.F $y = A\cos 2x + B\sin 2x$

P.I form $y = \lambda\cos x + \mu\sin x$

$\frac{dy}{dx} = -\lambda\sin x + \mu\cos x$

$\frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$

Substituting $(-\lambda\cos x - \mu\sin x) + 4(\lambda\cos x + \mu\sin x) = 6\sin x$

Terms in $\cos x$ $\quad -\lambda + 4\lambda = 0$

$\lambda = 0$

Terms in $\sin x$ $\quad -\mu + 4\mu = 6$

$\mu = 2$

General Solution.

$$y = A\cos 2x + B\sin 2x + 2\sin x$$

Particular Solution

when $y=2$, $x=0$ $\quad 2 = A(1) + B(0) + 2(0)$

$\therefore A = 2$

$y = 2\cos 2x + B\sin 2x + 2\sin x$

when $\frac{dy}{dx}=1$, $x=0$ $\quad \frac{dy}{dx} = -4\sin 2x + 2B\cos 2x + 2\cos x$

$1 = -4(0) + 2B(1) + 2(1)$

$\therefore B = -\frac{1}{2}$

$$y = 2\cos 2x - \frac{1}{2}\sin 2x + 2\sin x$$

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3 Aux Equation

$m^2 - 8m = 0$

$m(m-8) = 0$

$m=0$ or $m=8$

$\therefore$ C.F $\quad y = Ae^{0x} + Be^{8x}$

$y = A + Be^{8x}$

P.I $\quad$ Can't have a constant!

$y = \lambda x^2 + \mu x$

$y = \lambda x^2 + \mu x$

$\frac{dy}{dx} = 2\lambda x + \mu$

$\frac{d^2y}{dx^2} = 2\lambda$

Substituting in:

$$2\lambda - 8(2\lambda x + \mu) = 2x + 4$$

terms in x: $-16\lambda = 2$

$\lambda = -\frac{1}{8}$

Constants: $2\lambda - 8\mu = 4$

$-\frac{1}{4} - 8\mu = 4$

$-8\mu = \frac{17}{4}$

$\mu = -\frac{17}{32}$

$\therefore$ G.S $\quad y = A + Be^{8x} - \frac{1}{8}x^2 - \frac{17}{32}x$

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(4) Aux Eq: $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1$

$\therefore$ CF: $y = (A + Bx)e^x$

Particular solution.

$y = 6,\ x = 0 \quad \therefore \quad 6 = (A + 0)1$

$A = 6$

$$y = (6 + Bx)e^x$$

$\frac{dy}{dx} = -12,\ x = 0 \quad \frac{dy}{dx} = (6 + Bx)e^x + Be^x$

$-12 = (6)(1) + B$

$B = -18$

$\therefore \quad y = (6 - 18x)e^x \quad$ or $\quad y = 6(1 - 3x)e^x$

(5) Inspection on LHS as exact product derivative.

$$y\cot x = \int \sec^2 x \, dx$$

$$y\cot x = \tan x + c$$

$$y = \tan x(\tan x + c)$$

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(6) $$\frac{dy}{dx} = x - xy$$

$$\frac{dy}{dx} = x(1-y)$$

$$\int \frac{1}{1-y} dy = \int x \, dx$$

$$-\ln|1-y| = \frac{1}{2}x^2 + c$$

$$\ln|1-y| = -\frac{1}{2}x^2 + c$$

$$1 - y = e^{-\frac{1}{2}x^2 + c}$$

Loads of possible forms!

$$1 - y = Ae^{-\frac{1}{2}x^2}$$

$$y = 1 - Ae^{-\frac{1}{2}x^2} \quad \text{o.e.}$$

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(7) I.F $$e^{\int -\frac{1}{1+x}} = e^{-\ln|1+x|} = e^{\ln\left|\frac{1}{1+x}\right|} = \frac{1}{1+x}$$

Multiplying through:

$$\frac{1}{1+x}\frac{dy}{dx} - \frac{y}{(1+x)^2} = e^x$$

LHS exact product derivative.

$$\therefore \frac{y}{1+x} = \int e^x \, dx$$

$$\frac{y}{1+x} = e^x + c$$

when $x=0$, $y=5$

$$\frac{5}{1} = 1 + c \Rightarrow c = 4$$

$$y = (1+x)(e^x + 4) \quad \text{o.e}$$

⑧ Aux Equation

$$m^2 + m - 12 = 0$$
$$(m+4)(m-3) = 0$$
$$m = -4, m = 3$$

$\therefore$ CF $x = Ae^{-4t} + Be^{3t}$

P.I can't be $x = \lambda e^{3t}$ as part of the C.F

$\therefore x = \lambda te^{3t}$

$x = \lambda te^{3t}$

λt	λ
e^{3t}	$3e^{3t}$

Product rule.

$$\frac{dx}{dt} = 3\lambda te^{3t} + \lambda e^{3t}$$

$$\frac{d^2x}{dt^2} = 3\left[3\lambda te^{3t} + \lambda e^{3t}\right] + 3\lambda e^{3t}$$
$$= 9\lambda te^{3t} + 6\lambda e^{3t}$$

Substituting in

$$(9\lambda t + 6\lambda) + (3\lambda t + \lambda) - 12(\lambda t) = 14$$
$$7\lambda = 14$$
$$\therefore \lambda = 2$$

G.S $x = Ae^{-4t} + Be^{3t} + 2te^{3t}$

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⑨ Aux $m = -4 \pm 7i$

ⓐ $\therefore m^2 + 8m + 65 = 0$

$a = 8, b = 65$

ⓑ Need boundary conditions / Initial Conditions

(10) $x\frac{dy}{dx} + \frac{y}{\ln x} = \frac{x}{\ln x}$

(a) $\frac{dy}{dx} + \frac{y}{x\ln x} = \frac{1}{\ln x}$

$\ln x\frac{dy}{dx} + \frac{1}{x}y = 1$

LHS exact

$\therefore y\ln x = \int 1\,dx$

$y\ln x = x + c$

when $x = e$, $y = 1$

$\therefore (1)(1) = e + c$

$c = 1 - e$

$y\ln x = x + 1 - e$

$y = \frac{x + 1 - e}{\ln x}$

(b) For large values of x, $x > \ln x$ $\therefore$ as $x \to \infty$ $y \to \infty$.

(11) $\frac{d^2y}{dx^2} + y = 0$

(a) Aux equation

$m^2 + 1 = 0$

$m = \pm i$

$\therefore y = A\cos x + B\sin x$

(b) $y = \lambda x\cos x + \mu x\sin x$